

Strongly continuous orbit equivalence of one-sided topological Markov shifts

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Abstract

We will introduce a notion of strongly continuous orbit equivalence in one-sided topological Markov shifts. Strongly continuous orbit equivalence yields a topological conjugacy between their two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$. We prove that one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are strongly continuous orbit equivalent if and only if there exists an isomorphism between the Cuntz-Krieger algebras \mathcal{O}_A and \mathcal{O}_B preserving their maximal commutative C^* -subalgebras $C(X_A)$ and $C(X_B)$ and giving cocycle conjugate gauge actions. An example of one-sided topological Markov shifts which are strongly continuous orbit equivalent but not one-sided topologically conjugate is presented.

1 Introduction

Let $A = [A(i, j)]_{i, j=1}^N$ be an $N \times N$ matrix with entries in $\{0, 1\}$, where $1 < N \in \mathbb{N}$. Throughout the paper, we assume that A is irreducible and satisfies condition (I) in the sense of Cuntz–Krieger [2]. We denote by X_A the shift space

$$X_A = \{(x_n)_{n \in \mathbb{N}} \in \{1, \dots, N\}^{\mathbb{N}} \mid A(x_n, x_{n+1}) = 1 \text{ for all } n \in \mathbb{N}\} \quad (1.1)$$

of the right one-sided topological Markov shift for A . It is a compact Hausdorff space in natural product topology on $\{1, \dots, N\}^{\mathbb{N}}$. The shift transformation σ_A on X_A defined by $\sigma_A((x_n)_{n \in \mathbb{N}}) = (x_{n+1})_{n \in \mathbb{N}}$ is a continuous surjective map on X_A . The topological dynamical system (X_A, σ_A) is called the (right) one-sided topological Markov shift for A . The two-sided topological Markov shift written $(\bar{X}_A, \bar{\sigma}_A)$ is similarly defined by gathering two-sided sequences $(x_n)_{n \in \mathbb{Z}}$ instead of one-sided sequences $(x_n)_{n \in \mathbb{N}}$ in (1.1). In [2], J. Cuntz and W. Krieger have introduced a C^* -algebra from topological Markov shift (X_A, σ_A) . It is called the Cuntz–Krieger algebra written \mathcal{O}_A . They have proved that if one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are topologically conjugate, then the Cuntz–Krieger algebras \mathcal{O}_A and \mathcal{O}_B with their gauge actions are conjugate. They have also proved that if two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate, then the stabilized Cuntz–Krieger algebras $\mathcal{O}_A \otimes \mathcal{K}(H)$ and $\mathcal{O}_B \otimes \mathcal{K}(H)$ with their stabilized gauge actions are conjugate. We note that one-sided topological conjugacy of topological Markov shifts yields two-sided topological conjugacy. The author in [9] has introduced the notion of continuous orbit equivalence of one-sided topological

Markov shifts. It is a dynamical equivalence relation in one-sided topological Markov shifts inspired by studies of orbit equivalences in Cantor minimal systems by Giordano–Putnam–Skau (cf. [4], [5]), Giordano–Matui–Putnam–Skau (cf. [6]). It is weaker than one-sided topological conjugacy and gives rise to isomorphic Cuntz–Krieger algebras ([9]). Let A and B be irreducible square matrices with entries in $\{0, 1\}$. One-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are said to be continuously orbit equivalent if there exists a homeomorphism $h : X_A \rightarrow X_B$ such that

$$\sigma_B^{k_1(x)}(h(\sigma_A(x))) = \sigma_B^{l_1(x)}(h(x)) \quad \text{for } x \in X_A, \quad (1.2)$$

$$\sigma_A^{k_2(y)}(h^{-1}(\sigma_B(y))) = \sigma_A^{l_2(y)}(h^{-1}(y)) \quad \text{for } y \in X_B \quad (1.3)$$

for some continuous functions $k_1, l_1 \in C(X_A, \mathbb{Z}_+)$, $k_2, l_2 \in C(X_B, \mathbb{Z}_+)$. Let G_A denote the étale groupoid for (X_A, σ_A) whose reduced groupoid C^* -algebra $C_r^*(G_A)$ is isomorphic to the Cuntz–Krieger algebra \mathcal{O}_A (cf. [14], [15], [18]). Denote by \mathcal{D}_A the canonical maximal abelian C^* -subalgebra of \mathcal{O}_A realized as the commutative C^* -algebra of continuous functions on the unit space $G_A^{(0)}$ of G_A . The algebra \mathcal{D}_A is canonically isomorphic to the C^* -algebra $C(X_A)$ of continuous functions on the shift space X_A . H. Matui has studied continuous orbit equivalence from the view point of groupoids ([14], [15]).

In [12], we have obtained the following classification results of continuous orbit equivalence of one-sided topological Markov shifts.

Theorem 1.1 ([12], cf. [9], [10], [14], [15]). *The following four assertions are equivalent:*

- (i) (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent.
- (ii) The étale groupoids G_A and G_B are isomorphic.
- (iii) There exists an isomorphism $\Psi : \mathcal{O}_A \rightarrow \mathcal{O}_B$ such that $\Psi(\mathcal{D}_A) = \mathcal{D}_B$.
- (iv) \mathcal{O}_A and \mathcal{O}_B are isomorphic and $\det(\text{id} - A) = \det(\text{id} - B)$.

Let A be an irreducible square matrix with entries in $\{0, 1\}$. The ordered cohomology group (\bar{H}^A, \bar{H}_+^A) is defined by the quotient group of the ordered abelian group $C(\bar{X}_A, \mathbb{Z})$ of all \mathbb{Z} -valued continuous functions on \bar{X}_A quoted by the subgroup $\{\xi - \xi \circ \bar{\sigma}_A \mid \xi \in C(\bar{X}_A, \mathbb{Z})\}$. The positive cone \bar{H}_+^A consists of the classes of nonnegative functions in $C(\bar{X}_A, \mathbb{Z})$ (cf. [1], [17]). We similarly define the ordered cohomology group (H^A, H_+^A) for one-sided topological Markov shift (X_A, σ_A) . The latter ordered group (H^A, H_+^A) is naturally isomorphic to the former one (\bar{H}^A, \bar{H}_+^A) ([12, Lemma 3.1]). The ordered group (H^A, H_+^A) is also isomorphic to the first cohomology group $H^1(G_A, \mathbb{Z})$ of the groupoid G_A ([12, Proposition 3.4]). In [1], Boyle–Handelman have proved that the ordered cohomology group (\bar{H}^A, \bar{H}_+^A) is a complete invariant for flow equivalence of two-sided topological Markov shift $(\bar{X}_A, \bar{\sigma}_A)$.

In the first part of this paper, we will introduce a notion of continuous orbit map from (X_A, σ_A) to (X_B, σ_B) . A local homeomorphism $h : X_A \rightarrow X_B$ is said to be *continuous orbit map* if there exist continuous functions $k_1, l_1 : X_A \rightarrow \mathbb{Z}_+$ such that

$$\sigma_B^{k_1(x)}(h(\sigma_A(x))) = \sigma_B^{l_1(x)}(h(x)) \quad \text{for } x \in X_A. \quad (1.4)$$

It yields a morphism in the continuous orbit equivalence classes of one-sided topological Markov shifts. For $f \in C(X_B, \mathbb{Z})$, define

$$\Psi_h(f)(x) = \sum_{i=0}^{l_1(x)-1} f(\sigma_B^i(h(x))) - \sum_{j=0}^{k_1(x)-1} f(\sigma_B^j(h(\sigma_A(x)))) \quad \text{for } x \in X_A. \quad (1.5)$$

It is easy to see that $\Psi_h(f) \in C(X_A, \mathbb{Z})$. Thus $\Psi_h : C(X_B, \mathbb{Z}) \rightarrow C(X_A, \mathbb{Z})$ gives rise to a homomorphism of abelian groups and induces a homomorphism from H^B to H^A . We will then show that the objects of continuous orbit equivalence classes of one-sided topological Markov shifts with the morphisms of continuous orbit maps form a category (Proposition 2.6). We have

Theorem 1.2 (Theorem 3.10). *The correspondence Ψ yields a contravariant functor from the category of continuous orbit equivalence classes $[(X_A, \sigma_A)]$ of one-sided topological Markov shifts to that of ordered abelian groups (H^A, H_+^A) .*

The class $[1_A] \in H^A$ of the constant function $1_A(x) = 1, x \in X_A$ is an order unit of the ordered group (H^A, H_+^A) . Let $h : X_A \rightarrow X_B$ be a continuous orbit map giving rise to a continuous orbit equivalence between (X_A, σ_A) and (X_B, σ_B) . It is a key ingredient in the papers [12], [13] that the class $[\Psi_h(1_B)]$ in H^A of $\Psi_h(1_B) \in C(X_A, \mathbb{Z})$ belongs to the positive cone H_+^A . In the second part of this paper, we will introduce a notion of strongly continuous orbit equivalence between (X_A, σ_A) and (X_B, σ_B) , which is defined by the condition that $[\Psi_h(1_B)] = [1_A]$ in H^A . It has been proved in [13] that under the condition that $[\Psi_h(1_B)] = [1_A]$ in H^A , their zeta functions coincide, that is, $\det(\text{id} - tA) = \det(\text{id} - tB)$. Hence strongly continuous orbit equivalence preserves the structure of periodic points of their two-sided topological Markov shifts. We will know that strongly continuous orbit equivalence in one-sided topological Markov shifts is a subequivalence relation in continuous orbit equivalence, so that the objects of strongly continuous orbit equivalence classes of one-sided topological Markov shifts with the morphisms of strongly continuous orbit maps form a category (Proposition 4.5). Continuous orbit equivalence of one-sided topological Markov shifts does not necessarily give rise to topological conjugacy of their two-sided topological Markov shifts. We however see the following theorem:

Theorem 1.3 (Theorem 5.5 and Corollary 5.7). *Suppose that (X_A, σ_A) and (X_B, σ_B) are strongly continuous orbit equivalent. Then their two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate. Hence their C^* -crossed products are isomorphic:*

$$C(\bar{X}_A) \times_{\bar{\sigma}_A^*} \mathbb{Z} \cong C(\bar{X}_B) \times_{\bar{\sigma}_B^*} \mathbb{Z}.$$

Let us denote by ρ^A the gauge action on \mathcal{O}_A . In general, continuous orbit equivalence does not necessarily yield the cocycle conjugacy of the gauge actions on the Cuntz–Krieger algebras. We have the following result which is a generalization of [2, 2.17 Proposition].

Theorem 1.4 (Theorem 6.7). *The following two assertions are equivalent.*

- (i) *One-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are strongly continuous orbit equivalent.*

- (ii) *There exist a unitary one-cocycle $v_t \in U(\mathcal{O}_B)$, $t \in \mathbb{T}$ for the gauge action ρ^B on \mathcal{O}_B and an isomorphism $\Phi : \mathcal{O}_A \rightarrow \mathcal{O}_B$ such that*

$$\Phi(\mathcal{D}_A) = \mathcal{D}_B \quad \text{and} \quad \Phi \circ \rho_t^A = \text{Ad}(v_t) \circ \rho_t^B \circ \Phi, \quad t \in \mathbb{T}.$$

Hence if (X_A, σ_A) and (X_B, σ_B) are strongly continuous orbit equivalent, then the dual actions of the gauge actions on the Cuntz–Krieger algebras are isomorphic (Corollary 6.8):

$$(\mathcal{O}_A \rtimes_{\rho^A} \mathbb{T}, \hat{\rho}^A, \mathbb{Z}) \cong (\mathcal{O}_B \rtimes_{\rho^B} \mathbb{T}, \hat{\rho}^B, \mathbb{Z}).$$

One-sided topological conjugacy yields a strongly continuous orbit equivalence. We will finally present an example of one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) which are strongly continuous orbit equivalent but not topologically conjugate. Let A and B be the following matrices:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}. \quad (1.6)$$

They are both irreducible and satisfy condition (I). We will show the following theorem.

Theorem 1.5 (Theorem 7.1). *The one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) defined by the matrices (1.6) are strongly continuous orbit equivalent, but not topologically conjugate.*

Throughout the paper, we will use the following notations. The set of positive integers and the set of nonnegative integers are denoted by \mathbb{N} and by \mathbb{Z}_+ respectively. A word $\mu = \mu_1 \cdots \mu_k$ for $\mu_i \in \{1, \dots, N\}$ is said to be admissible for X_A if μ appears in somewhere in an element x in X_A . The length of μ is k , which is denoted by $|\mu|$. We denote by $B_k(X_A)$ the set of all admissible words of length k . We set $B_*(X_A) = \bigcup_{k=0}^{\infty} B_k(X_A)$ where $B_0(X_A)$ denotes the empty word \emptyset . Denote by U_μ the cylinder set $\{(x_n)_{n \in \mathbb{N}} \in X_A \mid x_1 = \mu_1, \dots, x_k = \mu_k\}$ for $\mu = \mu_1 \cdots \mu_k \in B_k(X_A)$. For $x = (x_n)_{n \in \mathbb{N}} \in X_A$ and $k, l \in \mathbb{N}$ with $k \leq l$, we set

$$x_{[k,l]} = x_k x_{k+1} \cdots x_l \in B_{l-k+1}(X_A), \quad x_{[k,\infty)} = (x_k, x_{k+1}, \dots) \in X_A.$$

We denote by $C(X_A, \mathbb{Z}_+)$ the set of \mathbb{Z}_+ -valued continuous functions on X_A . A point $x \in X_A$ is said to be eventually periodic if $\sigma_A^r(x) = \sigma_A^s(x)$ for some $r, s \in \mathbb{Z}_+$ with $r \neq s$.

2 Continuous orbit maps

Definition 2.1. Let (X_A, σ_A) and (X_B, σ_B) be one-sided topological Markov shifts. A local homeomorphism $h : X_A \rightarrow X_B$ is said to be *continuous orbit map* if there exist continuous functions $k_1, l_1 : X_A \rightarrow \mathbb{Z}_+$ such that

$$\sigma_B^{k_1(x)}(h(\sigma_A(x))) = \sigma_B^{l_1(x)}(h(x)) \quad \text{for } x \in X_A. \quad (2.1)$$

It is denoted by $h : (X_A, \sigma_A) \rightarrow (X_B, \sigma_B)$. If a continuous orbit map $h : (X_A, \sigma_A) \rightarrow (X_B, \sigma_B)$ is a homeomorphism such that its inverse $h^{-1} : (X_B, \sigma_B) \rightarrow (X_A, \sigma_A)$ is also a continuous orbit map, it is called a *continuous orbit homeomorphism*.

Hence (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent if and only if there exists a continuous orbit homeomorphism $h : (X_A, \sigma_A) \rightarrow (X_B, \sigma_B)$.

For a continuous orbit map $h : (X_A, \sigma_A) \rightarrow (X_B, \sigma_B)$ with continuous functions $k_1, l_1 : X_A \rightarrow \mathbb{Z}_+$ satisfying (2.1), we put for $n \in \mathbb{N}$,

$$k_1^n(x) = \sum_{i=0}^{n-1} k_1(\sigma_A^i(x)), \quad l_1^n(x) = \sum_{i=0}^{n-1} l_1(\sigma_A^i(x)) \quad \text{for } x \in X_A.$$

We note that the following identities hold.

Lemma 2.2 (cf. [13, Lemma 3.1]). *For $n, m \in \mathbb{Z}_+$, we have*

$$k_1^{n+m}(x) = k_1^n(x) + k_1^m(\sigma_A^n(x)) \quad \text{for } x \in X_A, \quad (2.2)$$

$$l_1^{n+m}(x) = l_1^n(x) + l_1^m(\sigma_A^n(x)) \quad \text{for } x \in X_A \quad (2.3)$$

and

$$\sigma_B^{k_1^n(x)}(h(\sigma_A^n(x))) = \sigma_B^{l_1^n(x)}(h(x)) \quad \text{for } x \in X_A. \quad (2.4)$$

Lemma 2.3. *Let A, B, C be irreducible matrices with entries in $\{0, 1\}$ satisfying condition (I). Let $h : (X_A, \sigma_A) \rightarrow (X_B, \sigma_B)$ and $g : (X_B, \sigma_B) \rightarrow (X_C, \sigma_C)$ be continuous orbit maps such that there exist continuous functions $k_1, l_1 : X_A \rightarrow \mathbb{Z}_+$ and $k_2, l_2 : X_B \rightarrow \mathbb{Z}_+$ satisfying*

$$\sigma_B^{k_1(x)}(h(\sigma_A(x))) = \sigma_B^{l_1(x)}(h(x)) \quad \text{for } x \in X_A, \quad (2.5)$$

$$\sigma_C^{k_2(y)}(g(\sigma_B(y))) = \sigma_C^{l_2(y)}(g(y)) \quad \text{for } y \in X_B. \quad (2.6)$$

Put

$$k_3(x) = k_2^{l_1(x)}(h(x)) + l_2^{k_1(x)}(h(\sigma_A(x))) \quad \text{for } x \in X_A, \quad (2.7)$$

$$l_3(x) = l_2^{l_1(x)}(h(x)) + k_2^{k_1(x)}(h(\sigma_A(x))) \quad \text{for } x \in X_A. \quad (2.8)$$

Then we have

$$\sigma_C^{k_3(x)}(g \circ h(\sigma_A(x))) = \sigma_C^{l_3(x)}(g \circ h(x)) \quad \text{for } x \in X_A. \quad (2.9)$$

Hence $g \circ h : X_A \rightarrow X_C$ gives rise to a continuous orbit map.

Proof. Take an arbitrary element $x \in X_A$. For $n \in \mathbb{N}$ and $y \in X_B$, we have by (2.4)

$$\sigma_C^{k_2^n(y)}(g(\sigma_B^n(y))) = \sigma_C^{l_2^n(y)}(g(y)). \quad (2.10)$$

Apply (2.10) for $n = l_1(x), y = h(x)$, one has

$$\sigma_C^{k_2^{l_1(x)}(h(x))}(g(\sigma_B^{l_1(x)}(h(x)))) = \sigma_C^{l_2^{l_1(x)}(h(x))}(g(h(x))).$$

Apply (2.10) for $n = k_1(x), y = h(\sigma_A(x))$, one has

$$\sigma_C^{k_2^{k_1(x)}(h(\sigma_A(x)))}(g(\sigma_B^{k_1(x)}(h(\sigma_A(x))))) = \sigma_C^{l_2^{k_1(x)}(h(\sigma_A(x)))}(g(h(\sigma_A(x)))).$$

Put $n = l_1(x), m = k_1(x)$. By (2.5), we have

$$\begin{aligned}
\sigma_C^{k_2^n(h(x)) + l_2^m(h(\sigma_A(x)))}(g \circ h(\sigma_A(x))) &= \sigma_C^{k_2^n(h(x))}(\sigma_C^{k_2^m(h(\sigma_A(x)))}(g(\sigma_B^m(h(\sigma_A(x))))) \\
&= \sigma_C^{k_2^n(h(x))}(\sigma_C^{k_2^m(h(\sigma_A(x)))}(g(\sigma_B^n(h(x))))) \\
&= \sigma_C^{k_2^m(h(\sigma_A(x)))}(\sigma_C^{l_2^n(h(x))}(g(h(x)))) \\
&= \sigma_C^{k_2^m(h(\sigma_A(x))) + l_2^n(h(x))}(g(h(x))).
\end{aligned}$$

□

More generally we have the following formula. The proof is routine.

Lemma 2.4. *Keep the above situation. For $p \in \mathbb{N}$, we have*

$$\sigma_C^{k_2^n(h(x)) + l_2^m(h(\sigma_A^p(x)))}(g \circ h(\sigma_A^p(x))) = \sigma_C^{k_2^m(h(\sigma_A^p(x))) + l_2^n(h(x))}(g(h(x)))$$

where $n = l_1^p(x), m = k_1^p(x)$.

We have the following proposition.

Proposition 2.5. *Let A, B, C, D be irreducible matrices with entries in $\{0, 1\}$ satisfying condition (I). Let $h : (X_A, \sigma_A) \rightarrow (X_B, \sigma_B)$ and $h' : (X_C, \sigma_C) \rightarrow (X_D, \sigma_D)$ be continuously orbit homeomorphisms. If there exists a continuous orbit map $g : (X_A, \sigma_A) \rightarrow (X_C, \sigma_C)$, then the map $h' \circ g \circ h^{-1} : X_B \rightarrow X_D$ becomes a continuous orbit map. Hence continuous orbit maps form morphisms of the continuous orbit equivalence classes of one-sided topological Markov shifts.*

Therefore we have

Proposition 2.6. *The objects of continuous orbit equivalence classes of one-sided topological Markov shifts with the morphisms of continuous orbit maps form a category.*

3 Cohomology groups

For a one-sided topological Markov shift (X_A, σ_A) , we denote by $\text{cobdy}(\sigma_A)$ the subgroup $\{\xi - \xi \circ \sigma_A \mid \xi \in C(X_A, \mathbb{Z})\}$ of $C(X_A, \mathbb{Z})$, and set the quotient group

$$H^A = C(X_A, \mathbb{Z}) / \{\xi - \xi \circ \sigma_A \mid \xi \in C(X_A, \mathbb{Z})\}.$$

In this section, we will construct a contravariant functor Ψ from the category of continuous orbit equivalence classes of one-sided topological Markov shifts to the category of abelian groups. Let $h : (X_A, \sigma_A) \rightarrow (X_B, \sigma_B)$ be a continuous orbit map with continuous functions $k_1, l_1 : X_A \rightarrow \mathbb{Z}$ satisfying (2.1).

Lemma 3.1. *The function $c_1(x) = l_1(x) - k_1(x)$ for $x \in X_A$ does not depend on the choice of the continuous functions k_1, l_1 satisfying (2.1).*

Proof. Let $k'_1, l'_1 \in C(X_A, \mathbb{Z})$ be another continuous functions for h satisfying

$$\sigma_B^{k'_1(x)}(h(\sigma_A(x))) = \sigma_B^{l'_1(x)}(h(x)) \quad \text{for } x \in X_A. \quad (3.1)$$

Since k_1, k'_1 are both continuous, there exists $K \in \mathbb{N}$ such that $k_1(x), k'_1(x) \leq K$ for all $x \in X_A$. Put $c'_1(x) = l'_1(x) - k'_1(x)$ so that we have

$$\sigma_B^{c_1(x)+K}(h(x)) = \sigma_B^K(h(\sigma_A(x))) = \sigma_B^{c'_1(x)+K}(h(x)) \quad \text{for } x \in X_A.$$

Suppose that $c_1(x_0) \neq c'_1(x_0)$ for some $x_0 \in X_A$. There exists a clopen neighborhood U of x_0 such that

$$c_1(x) \neq c'_1(x) \quad \text{for all } x \in U.$$

Now h is a local homeomorphism, one may take a clopen neighborhood $V \subset U$ of x_0 such that $h : V \rightarrow h(V)$ is a homeomorphism. As $c_1(x) + K \neq c'_1(x) + K$ for all $x \in V$, $h(x)$ are eventually periodic points for all $x \in V$, which is a contradiction to the fact that the set of non eventually periodic points of $h(V)$ is dense in $h(V)$. \square

We call the above function c_1 the cocycle function of h . For $f \in C(X_B, \mathbb{Z})$, define

$$\Psi_h(f)(x) = \sum_{i=0}^{l_1(x)-1} f(\sigma_B^i(h(x))) - \sum_{j=0}^{k_1(x)-1} f(\sigma_B^j(h(\sigma_A(x))))), \quad x \in X_A. \quad (3.2)$$

It is easy to see that $\Psi_h(f) \in C(X_A, \mathbb{Z})$. Thus $\Psi_h : C(X_B, \mathbb{Z}) \rightarrow C(X_A, \mathbb{Z})$ gives rise to a homomorphism of abelian groups.

Lemma 3.2. $\Psi_h : C(X_B, \mathbb{Z}) \rightarrow C(X_A, \mathbb{Z})$ does not depend on the choice of the functions k_1, l_1 satisfying (2.1).

Proof. The proof is similar to the proof of [13, Lemma 4.2]. \square

Examples.

1. Let $h : X_A \rightarrow X_B$ be a topological conjugacy as one-sided subshifts. Then h is a continuous orbit map such that $\Psi_h(f) = f \circ h$ for $f \in C(X_B, \mathbb{Z})$.

2. For $A = B$, the shift map $\sigma_A : X_A \rightarrow X_A$ is a continuous orbit map on X_A such that $\Psi_{\sigma_A}(f) = f \circ \sigma_A$ for $f \in C(X_A, \mathbb{Z})$.

The equalities in the following lemma are basic in our further discussions. The proof is similar to the proof of [13, Lemma 4.3].

Lemma 3.3. Let $h : (X_A, \sigma_A) \rightarrow (X_B, \sigma_B)$ be a continuous orbit map with continuous functions k_1, l_1 satisfying (2.1). For $f \in C(X_B, \mathbb{Z})$, $x \in X_A$ and $m = 1, 2, \dots$, the following equalities hold:

$$\begin{aligned} & \sum_{i=0}^{m-1} \left\{ \sum_{i'=0}^{l_1(\sigma_B^i(x))-1} f(\sigma_B^{i'}(h(\sigma_A^i(x)))) - \sum_{j'=0}^{k_1(\sigma_A^i(x))-1} f(\sigma_B^{j'}(h(\sigma_A^i(x)))) \right\} \\ &= \sum_{i'=0}^{l_1^m(x)-1} f(\sigma_B^{i'}(h(x))) - \sum_{j'=0}^{k_1^m(x)-1} f(\sigma_B^{j'}(h(\sigma_A^m(x)))). \end{aligned}$$

Let $h : (X_A, \sigma_A) \rightarrow (X_B, \sigma_B)$ and $g : (X_B, \sigma_B) \rightarrow (X_C, \sigma_C)$ be continuous orbit maps with continuous functions $k_1, l_1 \in C(X_A, \mathbb{Z})$ and $k_2, l_2 \in C(X_B, \mathbb{Z})$ satisfying (2.5) and (2.6), respectively. We write the continuous orbit map $g \circ h : X_A \rightarrow X_C$ as gh . We will prove the following proposition

Proposition 3.4. $\Psi_h \circ \Psi_g = \Psi_{gh}$.

To prove the proposition, we provide some notations and a lemma. Let $k_3, l_3 : X_A \rightarrow \mathbb{Z}_+$ be the continuous functions defined by (2.7), (2.8), respectively. By Lemma 2.3, we have for $f \in C(X_C, \mathbb{Z})$

$$\Psi_{gh}(f)(x) = \sum_{i=0}^{l_3(x)-1} f(\sigma_C^i(gh(x))) - \sum_{j=0}^{k_3(x)-1} f(\sigma_C^j(gh(\sigma_A(x))))), \quad x \in X_A. \quad (3.3)$$

Keep the above situations. By Lemma 3.3, we have

Lemma 3.5. For $f \in C(X_C, \mathbb{Z})$, $x \in X_A$ and $m = 1, 2, \dots$, the following equalities hold:

$$\begin{aligned} & \sum_{i=0}^{m-1} \left\{ \sum_{i'=0}^{l_2(\sigma_B^i(h(x)))-1} f(\sigma_C^{i'}(g(\sigma_B^i(h(x))))) - \sum_{j'=0}^{k_2(\sigma_B^i(h(x)))-1} f(\sigma_C^{j'}(g(\sigma_B^i(h(x))))) \right\} \\ &= \sum_{i'=0}^{l_2^m(h(x))-1} f(\sigma_C^{i'}(gh(x))) - \sum_{j'=0}^{k_2^m(h(x))-1} f(\sigma_C^{j'}(g(\sigma_B^m(h(x))))) \end{aligned}$$

Hence we have

Corollary 3.6. For $f \in C(X_C, \mathbb{Z})$ and $x \in X_A$, we have

(i)

$$\begin{aligned} & \sum_{i=0}^{l_1(x)-1} \left\{ \sum_{i'=0}^{l_2(\sigma_B^i(h(x)))-1} f(\sigma_C^{i'}(g(\sigma_B^i(h(x))))) - \sum_{j'=0}^{k_2(\sigma_B^i(h(x)))-1} f(\sigma_C^{j'}(g(\sigma_B^i(h(x))))) \right\} \\ &= \sum_{i'=0}^{l_2^{l_1(x)}(h(x))-1} f(\sigma_C^{i'}(gh(x))) - \sum_{j'=0}^{k_2^{l_1(x)}(h(x))-1} f(\sigma_C^{j'}(g(\sigma_B^{l_1(x)}(h(x))))) \end{aligned}$$

so that

$$\begin{aligned} & \sum_{i=0}^{l_1(x)-1} \Psi_g(f)(\sigma_B^i(h(x))) \\ &= \sum_{i'=0}^{l_2^{l_1(x)}(h(x))-1} f(\sigma_C^{i'}(gh(x))) - \sum_{j'=0}^{k_2^{l_1(x)}(h(x))-1} f(\sigma_C^{j'}(g(\sigma_B^{l_1(x)}(h(x))))) \end{aligned}$$

(ii)

$$\begin{aligned}
& \sum_{i=0}^{k_1(x)-1} \left\{ \sum_{i'=0}^{l_2(\sigma_B^i(h(\sigma_A(x))))-1} f(\sigma_C^{i'}(g(\sigma_B^i(h(\sigma_A(x))))) - \sum_{j'=0}^{k_2(\sigma_B^i(h(\sigma_A(x))))-1} f(\sigma_C^{j'}(g(\sigma_B^i(h(\sigma_A(x))))) \right\} \\
&= \sum_{i'=0}^{l_2^{k_1(x)}(h(\sigma_A(x)))-1} f(\sigma_C^{i'}(gh(\sigma_A(x)))) - \sum_{j'=0}^{k_2^{k_1(x)}(h(\sigma_A(x)))-1} f(\sigma_C^{j'}(g(\sigma_B^{k_1(x)}(h(\sigma_A(x)))))
\end{aligned}$$

so that

$$\begin{aligned}
& \sum_{j=0}^{k_1(x)-1} \Psi_g(f)(\sigma_B^j(h(\sigma_A(x)))) \\
&= \sum_{i'=0}^{l_2^{k_1(x)}(h(\sigma_A(x)))-1} f(\sigma_C^{i'}(gh(\sigma_A(x)))) - \sum_{j'=0}^{k_2^{k_1(x)}(h(\sigma_A(x)))-1} f(\sigma_C^{j'}(g(\sigma_B^{k_1(x)}(h(\sigma_A(x)))))
\end{aligned}$$

(Proof of Proposition 3.4)

By the previous corollary, we have

$$\begin{aligned}
& \Psi_h(\Psi_g(f))(x) \\
&= \sum_{i=0}^{l_1(x)-1} \Psi_g(f)(\sigma_B^i(h(x))) - \sum_{j=0}^{k_1(x)-1} \Psi_g(f)(\sigma_B^j(h(\sigma_A(x)))) \\
&= \left\{ \sum_{i'=0}^{l_2^{l_1(x)}(h(x))-1} f(\sigma_C^{i'}(gh(x))) - \sum_{j'=0}^{k_2^{l_1(x)}(h(x))-1} f(\sigma_C^{j'}(g(\sigma_B^{l_1(x)}(h(x))))) \right\} \\
&- \left\{ \sum_{i'=0}^{l_2^{k_1(x)}(h(\sigma_A(x)))-1} f(\sigma_C^{i'}(gh(\sigma_A(x)))) - \sum_{j'=0}^{k_2^{k_1(x)}(h(\sigma_A(x)))-1} f(\sigma_C^{j'}(g(\sigma_B^{k_1(x)}(h(\sigma_A(x))))) \right\}.
\end{aligned}$$

By (2.8), the first $\{ \cdot \}$ above goes to

$$\sum_{i'=0}^{l_3(x)-1} f(\sigma_C^{i'}(gh(x))) \tag{3.4}$$

$$-\left\{ \sum_{i'=l_2^{l_1(x)}(h(x))}^{l_3(x)-1} f(\sigma_C^{i'}(gh(x))) + \sum_{j'=0}^{k_2^{l_1(x)}(h(x))-1} f(\sigma_C^{j'}(g(\sigma_B^{l_1(x)}(h(x))))) \right\}. \tag{3.5}$$

By (2.7), the second $\{\cdot\}$ above goes to

$$\sum_{i'=0}^{k_3(x)-1} f(\sigma_C^{i'}(gh(\sigma_A(x)))) \quad (3.6)$$

$$-\left\{ \sum_{i'=l_2^{k_1(x)}(h(\sigma_A(x)))}^{k_3(x)-1} f(\sigma_C^{i'}(gh(\sigma_A(x)))) + \sum_{j'=0}^{k_2^{k_1(x)}(h(\sigma_A(x)))-1} f(\sigma_C^{j'}(g(\sigma_B^{k_1(x)}(h(\sigma_A(x)))))) \right\}. \quad (3.7)$$

We thus see

$$\Psi_h(\Psi_g(f))(x) = \{(3.4) - (3.5)\} - \{(3.6) - (3.7)\}.$$

Since $\sigma_C^{l_2^{l_1(x)}(h(x))}(gh(x)) = \sigma_C^{k_2^{l_1(x)}(h(x))}(g(\sigma_B^{l_1(x)}(h(x))))$, we have

$$\begin{aligned} & (3.5) \\ &= \sum_{j'=0}^{k_2^{k_1(x)}(h(\sigma_A(x)))-1} f(\sigma_C^{j'}(\sigma_C^{l_1(x)}(h(x)))(gh(x)))) + \sum_{j'=0}^{k_2^{l_1(x)}(h(x))-1} f(\sigma_C^{j'}(g(\sigma_B^{l_1(x)}(h(x)))))) \\ &= \sum_{j'=0}^{k_2^{k_1(x)}(h(\sigma_A(x)))-1} f(\sigma_C^{j'}(\sigma_C^{k_2^{l_1(x)}(h(x))}(g(\sigma_B^{l_1(x)}(h(x)))))) \\ &\quad + \sum_{j'=0}^{k_2^{l_1(x)}(h(x))-1} f(\sigma_C^{j'}(g(\sigma_B^{l_1(x)}(h(x)))))) \\ &= \sum_{j'=0}^{k_2^{l_1(x)}(h(x))+k_2^{k_1(x)}(h(\sigma_A(x)))-1} f(\sigma_C^{j'}(g(\sigma_B^{l_1(x)}(h(x)))))). \end{aligned}$$

Since $\sigma_C^{k_2^{k_1(x)}(h(\sigma_A(x)))}(gh(\sigma_A(x))) = \sigma_C^{k_2^{k_1(x)}(h(\sigma_A(x)))}(g(\sigma_B^{k_1(x)}(h(\sigma_A(x))))))$, we have

$$\begin{aligned} & (3.7) \\ &= \sum_{j'=0}^{k_2^{l_1(x)}(h(x))-1} f(\sigma_C^{j'}(\sigma_C^{l_2^{k_1(x)}(h(\sigma_A(x)))}(gh(\sigma_A(x)))))) + \sum_{j'=0}^{k_2^{k_1(x)}(h(\sigma_A(x)))-1} f(\sigma_C^{j'}(g(\sigma_B^{k_1(x)}(h(\sigma_A(x)))))) \\ &= \sum_{j'=0}^{k_2^{l_1(x)}(h(x))-1} f(\sigma_C^{j'}(\sigma_C^{k_2^{k_1(x)}(h(\sigma_A(x)))}(g(\sigma_B^{k_1(x)}(h(\sigma_A(x)))))) \\ &\quad + \sum_{j'=0}^{k_2^{k_1(x)}(h(\sigma_A(x)))-1} f(\sigma_C^{j'}(g(\sigma_B^{k_1(x)}(h(\sigma_A(x)))))) \\ &= \sum_{j'=0}^{k_2^{k_1(x)}(h(\sigma_A(x)))+k_2^{l_1(x)}(h(x))-1} f(\sigma_C^{j'}(g(\sigma_B^{k_1(x)}(h(\sigma_A(x)))))). \end{aligned}$$

As $\sigma_C^{j'}(g(\sigma_B^{k_1(x)}(h(\sigma_A(x)))) = \sigma_C^{j'}(g(\sigma_B^{l_1(x)}(h(x))))$, we have (3.5) = (3.7) so that

$$\Psi_h(\Psi_g(f))(x) = (3.4) - (3.6) = \Psi_{gh}(f)(x).$$

□

Corollary 3.7 ([13, Proposition 4.5]). *Let $h : (X_A, \sigma_A) \rightarrow (X_B, \sigma_B)$ be a continuous orbit homeomorphism. Then we have $\Psi_h \circ \Psi_{h^{-1}} = \text{id}_{C(X_A, \mathbb{Z})}$ and $\Psi_{h^{-1}} \circ \Psi_h = \text{id}_{C(X_B, \mathbb{Z})}$.*

Proof. Take C as A and g as h^{-1} in the preceding proposition. Since by [13, Lemma 3.3] or Lemma 5.3,

$$l_2^{l_1(x)}(h(x)) + k_2^{k_1(x)}(h(\sigma_A(x))) - 1 = l_2^{k_1(x)}(h(\sigma_A(x))) + k_2^{l_1(x)}(h(x)),$$

we have $l_3(x) - 1 = k_3(x)$ so that for $f \in C(X_A, \mathbb{Z})$ the equality (3.4) - (3.6) = $f(x)$ holds. Hence $(\Psi_h \circ \Psi_{h^{-1}})(f)(x) = f(x)$. Similarly we have $\Psi_{h^{-1}} \circ \Psi_h = \text{id}_{C(X_B, \mathbb{Z})}$. □

Lemma 3.8. *Let $h : (X_A, \sigma_A) \rightarrow (X_B, \sigma_B)$ be a continuous orbit map. Then we have*

$$\Psi_h(f - f \circ \sigma_B) = f \circ h - f \circ h \circ \sigma_A, \quad f \in C(X_B, \mathbb{Z}). \quad (3.8)$$

Proof. Since

$$\begin{aligned} \sum_{i=0}^{l_1(x)-1} (f - f \circ \sigma_B)(\sigma_B^i(h(x))) &= f(h(x)) - f(\sigma_B^{l_1(x)}(h(x))), \\ \sum_{j=0}^{k_1(x)-1} (f - f \circ \sigma_B)(\sigma_B^j(h(\sigma_A(x)))) &= f(h(\sigma_A(x))) - f(\sigma_B^{k_1(x)}(h(\sigma_A(x)))), \end{aligned}$$

the equality (3.8) follows from (2.1). □

Therefore we have

Proposition 3.9. *The homomorphism $\Psi_h : C(X_B, \mathbb{Z}) \rightarrow C(X_A, \mathbb{Z})$ induces a homomorphism of abelian groups $\bar{\Psi}_h : H^B \rightarrow H^A$.*

By a similar argument to the discussions of [13, Section 5], one may prove that Ψ_h preserves the positive cones of the ordered abelian groups, that is, $\bar{\Psi}_h(H_+^B) \subset H_+^A$. We will briefly state a machinery to prove $\bar{\Psi}_h(H_+^B) \subset H_+^A$. Following [13, Definition 5.5], an eventually periodic point $x \in X_A$ is said to be (r, s) -attracting for some $r, s \in \mathbb{Z}_+$ if it satisfies the following two conditions:

- (i) $\sigma_A^r(x) = \sigma_A^s(x)$.
- (ii) For any clopen neighborhood $W \subset X_A$ of x , there exist clopen sets $U, V \subset X_A$ and a homeomorphism $\varphi : V \rightarrow U$ such that
 - (a) $x \in U \subset V \subset W$.
 - (b) $\varphi(x) = x$.
 - (c) $\sigma_A^r(\varphi(w)) = \sigma_A^s(w)$ for all $w \in V$.

(d) $\lim_{n \rightarrow \infty} \varphi^n(w) = x$ for all $w \in V$.

Let $x \in X_A$ be an eventually periodic point. By [13, Lemma 5.6 and Lemma 5.7], there exists $r, s \in \mathbb{Z}_+$ such that x is (r, s) -attracting and hence $\sigma_A^r(x) = \sigma_A^s(x)$, $r > s$. Since $h : X_A \rightarrow X_B$ is a continuous orbit map, by using a similar argument to [13, Lemma 5.8 and Corollary 5.9], one may show that $h(x)$ is $(l_1^r(x) + k_1^s(x), k_1^r(x) + l_1^s(x))$ -attracting, and hence $l_1^r(x) + k_1^s(x) > k_1^r(x) + l_1^s(x)$. Put $r' = l_1^q(\sigma_A^s(x))$, $s' = k_1^q(\sigma_A^s(x))$ where $q = r - s$ and $z = \sigma_B^{l_1^s(x) + k_1^s(x)}(h(x)) \in X_B$. By [13, Lemma 5.3], we then have

$$r' - s' = (l_1^r(x) - l_1^s(x)) - (k_1^r(x) - k_1^s(x)) > 0 \quad \text{and} \quad \sigma_B^{r'}(z) = \sigma_B^{s'}(z).$$

We set for $f \in C(X_A, \mathbb{Z})$

$$\omega_f^{r,s}(x) = \sum_{i=0}^{r-1} f(\sigma_A^i(x)) - \sum_{j=0}^{s-1} f(\sigma_A^j(x)).$$

Then $[f]$ belongs to H_+^A if and only if $\omega_f^{r,s}(x) > 0$ ([12, Lemma 3.2], [13, Lemma 5.2]). By [13, Lemma 5.3], we have

$$\omega_{\Psi_h(f)}^{r,s}(x) = \omega_f^{r',s'}(z) \quad \text{for} \quad f \in C(X_B, \mathbb{Z}).$$

Hence $[f] \in H_+^B$ implies $\omega_{\Psi_h(f)}^{r,s}(x) = \omega_f^{r',s'}(z) > 0$ so that $[\Psi_h(f)]$ belongs to H_+^A .

We thus conclude

Theorem 3.10. *Let $h : (X_A, \sigma_A) \rightarrow (X_B, \sigma_B)$ and $g : (X_B, \sigma_B) \rightarrow (X_C, \sigma_C)$ be continuous orbit maps. Then the homomorphisms*

$$\Psi_h : C(X_B, \mathbb{Z}) \rightarrow C(X_A, \mathbb{Z}), \quad \Psi_g : C(X_C, \mathbb{Z}) \rightarrow C(X_B, \mathbb{Z})$$

satisfy the following conditions:

- (i) $\Psi_h \circ \Psi_g = \Psi_{g \circ h}$.
- (ii) $\Psi_h(\text{cobdy}(\sigma_B)) \subset \text{cobdy}(\sigma_A)$ and $\Psi_g(\text{cobdy}(\sigma_C)) \subset \text{cobdy}(\sigma_B)$.
- (iii) They induce homomorphisms $\bar{\Psi}_h : (H^B, H_+^B) \rightarrow (H^A, H_+^A)$ and $\bar{\Psi}_g : (H^C, H_+^C) \rightarrow (H^B, H_+^B)$ of ordered abelian groups such that $\bar{\Psi}_h \circ \bar{\Psi}_g = \bar{\Psi}_{g \circ h}$.

Corollary 3.11. *The correspondence $\bar{\Psi}$ gives rise to a contravariant functor from the category \mathcal{C}_{COE} of the continuous orbit equivalence classes of one-sided topological Markov shifts with continuous orbit maps as morphisms to the category \mathcal{A}_+ of ordered abelian groups:*

$$[(X_A, \sigma_A)] \in \mathcal{C}_{\text{COE}} \rightarrow (H^A, H_+^A) \in \mathcal{A}_+. \quad (3.9)$$

4 Strongly continuous orbit equivalence

Definition 4.1. A continuous orbit map $h : (X_A, \sigma_A) \rightarrow (X_B, \sigma_B)$ is called a *strongly continuous orbit map* if there exists a continuous function $b_1 : X_A \rightarrow \mathbb{Z}$ such that

$$\Psi_h(1_B)(x) = 1 + b_1(x) - b_1(\sigma_A(x)), \quad x \in X_A. \quad (4.1)$$

For a nonnegative integer N_1 , the function $b'_1(x) = b_1(x) + N_1$ still satisfies the above equality. One may assume that the function b_1 in (4.1) is nonnegative. If a continuous orbit homeomorphism is a strongly continuous orbit map, it is called a *strongly continuous orbit homeomorphism*.

Definition 4.2. One-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are said to be *strongly continuous orbit equivalent* if there exists a strongly continuous orbit homeomorphism $h : (X_A, \sigma_A) \rightarrow (X_B, \sigma_B)$ such that its inverse $h^{-1} : (X_B, \sigma_B) \rightarrow (X_A, \sigma_A)$ is also a strongly continuous orbit homeomorphism. In this case, we write $(X_A, \sigma_A) \underset{\text{SCOE}}{\sim} (X_B, \sigma_B)$.

By definition, (X_A, σ_A) and (X_B, σ_B) are strongly continuous orbit equivalent if and only if there exist a homeomorphism $h : X_A \rightarrow X_B$, continuous functions $k_1, l_1, b_1 : X_A \rightarrow \mathbb{Z}_+$ and $k_2, l_2, b_2 : X_B \rightarrow \mathbb{Z}_+$ such that

$$\sigma_B^{k_1(x)}(h(\sigma_A(x))) = \sigma_B^{l_1(x)}(h(x)), \quad x \in X_A, \quad (4.2)$$

$$\sigma_A^{k_2(y)}(h^{-1}(\sigma_B(y))) = \sigma_A^{l_2(y)}(h^{-1}(y)), \quad y \in X_B, \quad (4.3)$$

and

$$l_1(x) - k_1(x) = 1 + b_1(x) - b_1(\sigma_A(x)), \quad x \in X_A, \quad (4.4)$$

$$l_2(y) - k_2(y) = 1 + b_2(y) - b_2(\sigma_B(y)), \quad y \in X_B. \quad (4.5)$$

Recall that the cocycle functions c_1, c_2 are defined by $c_1(x) = l_1(x) - k_1(x)$ for $x \in X_A$ and $c_2(y) = l_2(y) - k_2(y)$ for $y \in X_B$.

Lemma 4.3. Suppose that $h : X_A \rightarrow X_B$ is a homeomorphism which gives rise to a continuous orbit equivalence between (X_A, σ_A) and (X_B, σ_B) . Then the following three conditions are equivalent:

- (i) $(X_A, \sigma_A) \underset{\text{SCOE}}{\sim} (X_B, \sigma_B)$.
- (ii) $[c_1] = [1_A] \in H^A$.
- (iii) $[c_2] = [1_B] \in H^B$.

Proof. (ii) \Rightarrow (iii). Suppose that $[c_1] = [1_A] \in H^A$. Take a continuous function $b_1 \in C(X_A, \mathbb{Z}_+)$ such that $c_1(x) = 1_A(x) + b_1(x) - b_1(\sigma_A(x))$, $x \in X_A$. Since $c_1 = \Psi_h(1_B)$, $c_2 = \Psi_{h^{-1}}(1_A)$, we have $\Psi_{h^{-1}}(c_1) = \Psi_{h^{-1}}(\Psi_h(1_B)) = 1_B$ so that

$$c_2 = \Psi_{h^{-1}}(c_1 - b_1 + b_1 \circ \sigma_A) = 1_B - \{\Psi_{h^{-1}}(b_1) - \Psi_{h^{-1}}(b_1) \circ \sigma_B\}.$$

This implies that $[c_2] = [1_B] \in H^B$. (iii) \Rightarrow (ii) is similar. It is clear that (i) is equivalent to the both conditions (ii) and (iii). \square

Therefore we have

Proposition 4.4. *Strongly continuous orbit equivalence is an equivalence relation in the set of one-sided topological Markov shifts.*

Proof. Let $h : (X_A, \sigma_A) \rightarrow (X_B, \sigma_B)$ and $g : (X_B, \sigma_B) \rightarrow (X_C, \sigma_C)$ be strongly continuous orbit homeomorphisms. Put $c_{AB} = \Psi_h(1_B)$, $c_{BC} = \Psi_g(1_C)$ the cocycle functions for h and g respectively. By the previous lemma, we know that the composition $g \circ h : X_A \rightarrow X_C$ yields a continuous orbit equivalence between (X_A, σ_A) and (X_C, σ_C) . Put $c_{AC} = \Psi_{g \circ h}(1_C)$ so that $c_{AC} = \Psi_h(\Psi_g(1_C)) = \Psi_h(c_{BC})$. Since $[c_{AB}] = [1_A]$, $[c_{BC}] = [1_B]$, one has $[c_{AC}] = [\Psi_h(1_B)] = [c_{AB}] = [1_A]$ so that $(X_A, \sigma_A) \underset{\text{SCOE}}{\sim} (X_C, \sigma_C)$. \square

Proposition 4.5. *The objects $\mathcal{O}_{\text{SCOE}}$ of strongly continuous orbit equivalence classes of one-sided topological Markov shifts with the morphisms $\mathcal{M}_{\text{SCOE}}$ of strongly continuous orbit maps form a category $\mathcal{C}_{\text{SCOE}} = (\mathcal{O}_{\text{SCOE}}, \mathcal{M}_{\text{SCOE}})$.*

5 Two-sided conjugacy

Throughout this section, we assume that $h : X_A \rightarrow X_B$ is a homeomorphism which gives rise to a strongly continuous orbit equivalence between (X_A, σ_A) and (X_B, σ_B) . Let $k_1, l_1, b_1 : X_A \rightarrow \mathbb{Z}_+$ and $k_2, l_2, b_2 : X_B \rightarrow \mathbb{Z}_+$ be continuous functions satisfying (4.2), (4.4) and (4.3), (4.5), respectively.

Lemma 5.1. *Put $\varphi_{b_1}(x) = \sigma_B^{b_1(x)}(h(x))$ for $x \in X_A$. Then we have*

$$\varphi_{b_1}(\sigma_A(x)) = \sigma_B(\varphi_{b_1}(x)), \quad x \in X_A. \quad (5.1)$$

Proof. It follows that

$$\begin{aligned} \varphi_{b_1}(\sigma_A(x)) &= \sigma_B^{b_1(\sigma_A(x))}(h(\sigma_A(x))) \\ &= \sigma_B^{1+b_1(x)-l_1(x)} \sigma_B^{k_1(x)}(h(\sigma_A(x))) \\ &= \sigma_B(\sigma_B^{b_1(x)}(h(x))) \\ &= \sigma_B(\varphi_{b_1}(x)). \end{aligned}$$

\square

For $n \in \mathbb{Z}_+$, put $c_1^n(x) = l_1^n(x) - k_1^n(x)$, $x \in X_A$ and $c_2^n(y) = l_2^n(y) - k_2^n(y)$, $y \in X_B$.

Lemma 5.2. *Keep the above notations.*

(i) $c_1^n(x) = n + b_1(x) - b_1(\sigma_A^n(x))$ for $x \in X_A$.

(ii) $c_2^n(y) = n + b_2(y) - b_2(\sigma_B^n(y))$ for $y \in X_B$.

Proof. (i) As $l_1(\sigma_A^m(x)) - k_1(\sigma_A^m(x)) = 1 + b_1(\sigma_A^m(x)) - b_1(\sigma_A^{m+1}(x))$, we have

$$c_1^n(x) = \sum_{m=0}^{n-1} l_1(\sigma_A^m(x)) - \sum_{m=0}^{n-1} k_1(\sigma_A^m(x)) = n + b_1(x) - b_1(\sigma_A^n(x)).$$

(ii) is similar to (i). \square

Since (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent, the following identities hold. Its proof is seen in [13, Lemma 3.3].

Lemma 5.3 ([13, Lemma 3.3]). *For $x \in X_A$, $y \in X_B$ and $p \in \mathbb{Z}_+$, we have*

- (i) $k_2^{l_1^p(x)}(h(x)) + l_2^{k_1^p(x)}(h(\sigma_A^p(x))) + p = k_2^{k_1^p(x)}(h(\sigma_A^p(x))) + l_2^{l_1^p(x)}(h(x)).$
- (ii) $k_1^{l_2^p(y)}(h^{-1}(y)) + l_1^{k_2^p(y)}(h^{-1}(\sigma_B^p(y))) + p = k_1^{k_2^p(y)}(h^{-1}(\sigma_B^p(y))) + l_1^{l_2^p(y)}(h^{-1}(y)).$

Keep the situation.

Lemma 5.4. *There exists $N_h \in \mathbb{N}$ such that $b_1(x) + b_2(h(x)) = N_h$ for all $x \in X_A$, and equivalently $b_2(y) + b_1(h^{-1}(y)) = N_h$ for all $y \in X_B$.*

Proof. By Lemma 5.3, we have

$$k_2^{l_1(x)}(h(x)) + l_2^{k_1(x)}(h(\sigma_A(x))) + 1 = k_2^{k_1(x)}(h(\sigma_A(x))) + l_2^{l_1(x)}(h(x))$$

so that

$$c_2^{k_1(x)}(h(\sigma_A(x))) + 1 = c_2^{l_1(x)}(h(x))$$

and hence

$$k_1(x) + b_2(h(\sigma_A(x))) - b_2(\sigma_B^{k_1(x)}(h(\sigma_A(x)))) + 1 = l_1(x) + b_2(h(x)) - b_2(\sigma_B^{l_1(x)}(h(x))).$$

As $\sigma_B^{k_1(x)}(h(\sigma_A(x))) = \sigma_B^{l_1(x)}(h(x))$, we have

$$k_1(x) + b_2(h(\sigma_A(x))) + 1 = l_1(x) + b_2(h(x))$$

so that

$$c_1(x) = b_2(h(\sigma_A(x))) - b_2(h(x)) + 1.$$

Hence we have

$$b_1(x) - b_1(\sigma_A(x)) = b_2(h(\sigma_A(x))) - b_2(h(x)).$$

This implies that the function $x \in X_A \rightarrow b_1(x) + b_2(h(x)) \in \mathbb{N}$ is σ_A -invariant, so that it is constant. \square

Theorem 5.5. *Suppose that $(X_A, \sigma_A) \underset{\text{SCOE}}{\sim} (X_B, \sigma_B)$. Then their two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate.*

Proof. By the preceding lemma, the map $\varphi_{b_1} : X_A \rightarrow X_B$ defined by $\varphi_{b_1}(x) = \sigma_B^{b_1(x)}(h(x))$ satisfies

$$\varphi_{b_1}(\sigma_A(x)) = \sigma_B(\varphi_{b_1}(x)), \quad x \in X_A. \quad (5.2)$$

For $\bar{x} = (x_i)_{i \in \mathbb{Z}} \in \bar{X}_A$ and $j \in \mathbb{Z}$, put $\bar{x}(j) = x_{[j, \infty)} \in X_A$. We set $\bar{y}[j] = \varphi_{b_1}(\bar{x}(j)) \in X_B$. It then follows that

$$\sigma_B(\bar{y}[j]) = \varphi_{b_1}(\sigma_A(\bar{x}(j))) = \varphi_{b_1}(\bar{x}(j+1)) = \bar{y}[j+1].$$

Hence we may define an element $\bar{y} = (y_j)_{j \in \mathbb{Z}} \in \bar{X}_B$ such that $\bar{y}_{[j, \infty)} = \bar{y}[j]$. We set $\bar{h}(\bar{x}) = \bar{y}$ so that $\bar{h} : \bar{X}_A \rightarrow \bar{X}_B$ is a continuous map. Since $(\bar{\sigma}_A(\bar{x}))(j) = x_{[j+1, \infty)} = \sigma_A(\bar{x}(j))$, we have

$$[\bar{h}(\bar{\sigma}_A(\bar{x}))]_{[j, \infty)} = \varphi_{b_1}([\bar{\sigma}_A(\bar{x})](j)) = \varphi_{b_1}(\sigma_A(\bar{x}(j))) = [\bar{h}(\bar{x})]_{[j+1, \infty)} = [\bar{\sigma}_B(\bar{h}(\bar{x}))]_{[j, \infty)}$$

so that

$$\bar{h}(\bar{\sigma}_A(\bar{x})) = \bar{\sigma}_B(\bar{h}(\bar{x})), \quad \bar{x} \in \bar{X}_A.$$

Hence $\bar{h} : \bar{X}_A \rightarrow \bar{X}_B$ is a sliding block code (see [8] for the definition of the sliding block code). One may similarly construct a sliding block code $\bar{h}^{-1} : \bar{X}_B \rightarrow \bar{X}_A$ from the inverse $h^{-1} : X_B \rightarrow X_A$ of h . We denote by $\psi_{b_2} : X_B \rightarrow X_A$ the continuous map defined by $\psi_{b_2}(y) = \sigma_A^{b_2(y)}(h^{-1}(y))$, which satisfies $\psi_{b_2}(\sigma_B(y)) = \sigma_A(\psi_{b_2}(y))$, $y \in X_B$. Then the map $\bar{h}^{-1} : \bar{X}_B \rightarrow \bar{X}_A$ satisfies the equality $(\bar{h}^{-1}(\bar{y}))_{[j, \infty)} = \psi_{b_2}(\bar{y}(j))$ for $j \in \mathbb{Z}$. It then follows that for $j \in \mathbb{Z}$

$$\begin{aligned} (\bar{h}^{-1}(\bar{h}(\bar{x})))_{[j, \infty)} &= \psi_{b_2}(\varphi_{b_1}(\bar{x}(j))) \\ &= \psi_{b_2}(\sigma_B^{b_1(\bar{x}(j))}(h(\bar{x}(j)))) \\ &= \sigma_A^{b_1(\bar{x}(j))}(\psi_{b_2}(h(\bar{x}(j)))) \\ &= \sigma_A^{b_1(\bar{x}(j))}(\sigma_A^{b_2(h(\bar{x}(j)))}(h^{-1}(h(\bar{x}(j))))) \\ &= \sigma_A^{b_1(\bar{x}(j)) + b_2(h(\bar{x}(j)))}(\bar{x}(j)) \\ &= (\bar{\sigma}_A^{b_1(\bar{x}(j)) + b_2(h(\bar{x}(j)))}(\bar{x}))_{[j, \infty)}. \end{aligned}$$

Take a constant number N_h in the preceding lemma so that we have

$$(\bar{h}^{-1}(\bar{h}(\bar{x})))_{[j, \infty)} = (\bar{\sigma}_A^{N_h}(\bar{x}))_{[j, \infty)} \quad \text{for all } j \in \mathbb{Z}$$

and hence

$$\bar{h}^{-1}(\bar{h}(\bar{x})) = \bar{\sigma}_A^{N_h}(\bar{x}) \quad \text{for all } \bar{x} \in \bar{X}_A.$$

We thereby know that $\bar{h} : \bar{X}_A \rightarrow \bar{X}_B$ is injective. Similarly we see

$$\bar{h}(\bar{h}^{-1}(\bar{y})) = \bar{\sigma}_B^{N_h}(\bar{y}) \quad \text{for all } \bar{y} \in \bar{X}_B$$

so that $\bar{h} : \bar{X}_A \rightarrow \bar{X}_B$ is surjective and gives rise to a topological conjugacy between $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$. \square

Corollary 5.6. *Suppose that $(X_A, \sigma_A) \underset{\text{SCOE}}{\sim} (X_B, \sigma_B)$. Then each is a finite factor of the other. In particular, (X_A, σ_A) and (X_B, σ_B) are weakly conjugate.*

Proof. Since the two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate, the assertion that each of their one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) is a finite factor of the other comes from a general theory of symbolic dynamics ([7, Exercise 2]). One also knows it from the equalities: $\psi_{b_2} \circ \varphi_{b_1} = \sigma_A^{N_h}$ and $\varphi_{b_1} \circ \psi_{b_2} = \sigma_B^{N_h}$. \square

Let us denote by $C(\bar{X}_A)$ the commutative C^* -algebra of complex valued continuous functions on \bar{X}_A . The homeomorphism $\bar{\sigma}_A$ on \bar{X}_A naturally induces an automorphism $\bar{\sigma}_A^*$ on $C(\bar{X}_A)$ by $\bar{\sigma}_A^*(f) = f \circ \bar{\sigma}_A^{-1}$ for $f \in C(\bar{X}_A)$.

Corollary 5.7. *Suppose that $(X_A, \sigma_A) \underset{\text{SCOE}}{\sim} (X_B, \sigma_B)$. Then their C^* -crossed products are isomorphic:*

$$C(\bar{X}_A) \times_{\bar{\sigma}_A^*} \mathbb{Z} \cong C(\bar{X}_B) \times_{\bar{\sigma}_B^*} \mathbb{Z}.$$

We note that K_0 -group $K_0(C(\bar{X}_A) \times_{\bar{\sigma}_A^*} \mathbb{Z})$ of the C^* -algebra $C(\bar{X}_A) \times_{\bar{\sigma}_A^*} \mathbb{Z}$ is isomorphic to the ordered group (\bar{H}^A, \bar{H}_+^A) (see [1, Theorem 5.2], [17, Remark 3.10]).

Let $\pi_A : \bar{X}_A \rightarrow X_A$ denote the restriction defined by $\pi_A((x_n)_{n \in \mathbb{Z}}) = (x_n)_{n \in \mathbb{N}} \in X_A$ for $(x_n)_{n \in \mathbb{Z}} \in \bar{X}_A$. The following proposition is a converse to Theorem 5.5.

Proposition 5.8. *Let $h : X_A \rightarrow X_B$ be a homeomorphism such that there exist a topological conjugacy $\bar{h} : (\bar{X}_A, \bar{\sigma}_A) \rightarrow (\bar{X}_B, \bar{\sigma}_B)$ as two-sided subshifts and continuous functions $f_1 : X_A \rightarrow \mathbb{Z}_+$, $f_2 : X_B \rightarrow \mathbb{Z}_+$ such that*

$$\begin{aligned} \pi_B(\bar{h}(\bar{x})) &= \sigma_B^{f_1(x)}(h(x)) \quad \text{for } \bar{x} \in \bar{X}_A, \\ \pi_A(\bar{h}^{-1}(\bar{y})) &= \sigma_A^{f_2(y)}(h^{-1}(y)) \quad \text{for } \bar{y} \in \bar{X}_B \end{aligned}$$

where $x = \pi_A(\bar{x})$, $y = \pi_B(\bar{y})$. Then $h : X_A \rightarrow X_B$ gives rise to a strongly continuous orbit homeomorphism. Hence we have $(X_A, \sigma_A) \underset{\text{SCOE}}{\sim} (X_B, \sigma_B)$.

Proof. We note that $\bar{h} \circ \bar{\sigma}_A = \bar{\sigma}_B \circ \bar{h}$ and $\pi_B \circ \bar{\sigma}_B = \sigma_B \circ \pi_B$. For $\bar{x} \in \bar{X}_A$, we have

$$\pi_B(\bar{h}(\bar{\sigma}_A(\bar{x}))) = \sigma_B(\pi_B(\bar{h}(\bar{x}))) = \sigma_B^{f_1(x)+1}(h(x)).$$

As $\pi_A(\bar{\sigma}_A(\bar{x})) = \sigma_A(x)$, we also have

$$\pi_B(\bar{h}(\bar{\sigma}_A(\bar{x}))) = \sigma_B^{f_1(\pi_A(\bar{\sigma}_A(\bar{x})))}(h(\pi_A(\bar{\sigma}_A(\bar{x})))) = \sigma_B^{f_1(\sigma_A(x))}(h(\sigma_A(x)))$$

so that

$$\sigma_B^{f_1(x)+1}(h(x)) = \sigma_B^{f_1(\sigma_A(x))}(h(\sigma_A(x))), \quad x \in X_A.$$

This implies that $h : X_A \rightarrow X_B$ and similarly $h^{-1} : X_B \rightarrow X_A$ give rise to strongly continuous orbit maps, which are the inverses to each other. \square

6 Cocycle conjugacy

Let us denote by S_1, \dots, S_N the generating partial isometries of the Cuntz-Krieger algebra \mathcal{O}_A satisfying

$$\sum_{j=1}^N S_j S_j^* = 1, \quad S_i^* S_i = \sum_{j=1}^N A(i, j) S_j S_j^*, \quad i = 1, \dots, N. \quad (6.1)$$

For $t \in \mathbb{R}/\mathbb{Z} = \mathbb{T}$, the correspondence $S_i \rightarrow e^{2\pi\sqrt{-1}t} S_i$ gives rise to an automorphism of \mathcal{O}_A which we denote by $\rho_t^A \in \text{Aut}(\mathcal{O}_A)$. The automorphisms yield an action of \mathbb{T} to $\text{Aut}(\mathcal{O}_A)$

which we call the gauge action. Let us denote by \mathcal{D}_A the C^* -subalgebra of \mathcal{O}_A generated by the projections of the form: $S_{i_1} \cdots S_{i_n} S_{i_n}^* \cdots S_{i_1}^*$, which is canonically isomorphic to the commutative C^* -algebra $C(X_A)$ by identifying the projection $S_{i_1} \cdots S_{i_n} S_{i_n}^* \cdots S_{i_1}^*$ with the characteristic function $\chi_{U_{i_1 \dots i_n}} \in C(X_A)$ of the cylinder set $U_{i_1 \dots i_n}$ for the word $i_1 \cdots i_n$.

Throughout the section, we assume that $(X_A, \sigma_A) \underset{\text{SCOE}}{\sim} (X_B, \sigma_B)$ and fix a strongly continuous orbit homeomorphism $h : (X_A, \sigma_A) \rightarrow (X_B, \sigma_B)$ and continuous functions $k_1, l_1, b_1 : X_A \rightarrow \mathbb{Z}_+$ and $k_2, l_2, b_2 : X_B \rightarrow \mathbb{Z}_+$ satisfying (4.2), (4.4) and (4.3), (4.5), respectively. By Lemma 5.4, we have the following lemma.

Lemma 6.1.

- (i) $b_1(x) - b_1(\sigma_A(x)) = -b_2(h(x)) + b_2(h(\sigma_A(x)))$ for $x \in X_A$.
- (ii) $b_2(y) - b_2(\sigma_B(y)) = -b_1(h^{-1}(y)) + b_1(h^{-1}(\sigma_B(y)))$ for $y \in X_B$.

For $i \in \{1, \dots, N\}$ and $x = (x_n)_{n \in \mathbb{N}} \in X_A$, we write $ix = (i, x_1, x_2, \dots)$.

Lemma 6.2. *For $i \in \{1, 2, \dots, N\}$ and $y \in X_B$ satisfying $ih^{-1}(y) \in X_A$, put $z = ih^{-1}(y)$. Then we have*

$$b_1(z) - b_1(\sigma_A(z)) = b_2(y) - b_2(h(z)). \quad (6.2)$$

Proof. Since $h(\sigma_A(z)) = y$, the desired equality follows from Lemma 6.1 (i). \square

Recall that ρ^B stands for the gauge action on \mathcal{O}_B . Denote by $U(\mathcal{O}_B)$ and $U(\mathcal{D}_B)$ the group of unitaries of \mathcal{O}_B and that of \mathcal{D}_B respectively. A continuous map $t \in \mathbb{R}/\mathbb{Z} = \mathbb{T} \rightarrow v_t \in U(\mathcal{O}_B)$ is called a one-cocycle for ρ^B if it satisfies $v_{t+s} = v_t \rho_t^B(v_s)$, $t, s \in \mathbb{T}$. Then the map $t \in \mathbb{T} \rightarrow \text{Ad}(v_t) \circ \rho_t^B \in \text{Aut}(\mathcal{O}_B)$ yields an action called a perturbed action of ρ^B by v . Since the function b_2 is regarded as a positive element of \mathcal{D}_B , one may define unitaries $u_t^{b_2} = \exp(2\pi\sqrt{-1}tb_2) \in U(\mathcal{D}_B)$, $t \in \mathbb{T}$. As $\rho_s^B(u_t^{b_2}) = u_t^{b_2}$ for all $s, t \in \mathbb{T}$, the family $\{u_t^{b_2}\}_{t \in \mathbb{T}}$ is a one-cocycle for ρ^B .

The following proposition is a generalization of [2, 2.17 Proposition].

Proposition 6.3. *Suppose that $(X_A, \sigma_A) \underset{\text{SCOE}}{\sim} (X_B, \sigma_B)$. Then there exists an isomorphism $\Phi : \mathcal{O}_A \rightarrow \mathcal{O}_B$ such that*

$$\Phi(\mathcal{D}_A) = \mathcal{D}_B \quad \text{and} \quad \Phi \circ \rho_t^A = \text{Ad}(u_t^{b_2}) \circ \rho_t^B \circ \Phi, \quad t \in \mathbb{T}.$$

Proof. The proof below follows essentially the proof of [9, Proposition 5.6.]. For the sake of completeness, we will give the proof in the following way. Let us denote by \mathfrak{H}_A (resp. \mathfrak{H}_B) the Hilbert space with its complete orthonormal system $\{e_x^A \mid x \in X_A\}$ (resp. $\{e_y^B \mid y \in X_B\}$). Consider the partial isometries $S_i^A, i = 1, \dots, N$ on \mathfrak{H}_A defined by

$$S_i^A e_x^A = \begin{cases} e_{ix}^A & \text{if } ix \in X_A, \\ 0 & \text{otherwise.} \end{cases} \quad (6.3)$$

Then the operators $S_i^A, i = 1, \dots, N$ are partial isometries satisfying the relations (6.1). For the $M \times M$ matrix $B = [B(i, j)]_{i, j=1}^M$, we similarly define the partial isometries $S_i^B, i = 1, \dots, M$ on \mathfrak{H}_B satisfying the relations (6.1) for B . Hence one may identify the Cuntz-Krieger algebra \mathcal{O}_A (resp. \mathcal{O}_B) with the C^* -algebra $C^*(S_1^A, \dots, S_N^A)$ (resp.

$C^*(S_1^B, \dots, S_M^B)$ generated by the partial isometries S_1^A, \dots, S_N^A (resp. S_1^B, \dots, S_M^B). For the continuous function $k_1 : X_A \rightarrow \mathbb{Z}_+$, let $K_1 = \text{Max}\{k_1(x) \mid x \in X_A\}$. By adding $K_1 - k_1(x)$ to $k_1(x)$ and $l_1(x)$, one may assume that $k_1(x) = K_1$ for all $x \in X_A$. Define the unitary $U_h : \mathfrak{H}_A \rightarrow \mathfrak{H}_B$ by $U_h e_x^A = e_{h(x)}^B$ for $x \in X_A$. We will see that $\Phi = \text{Ad}(U_h)$ satisfies the desired properties. We fix $i \in \{1, \dots, N\}$ and set $X_B^{(i)} = \{y \in X_B \mid ih^{-1}(y) \in X_A\}$. For $y \in X_B$, one has

$$U_h S_i^A U_h^* e_y^B = \begin{cases} e_{h(ih^{-1}(y))}^B & \text{if } y \in X_B^{(i)}, \\ 0 & \text{otherwise.} \end{cases}$$

For $y \in X_B^{(i)}$, put $z = ih^{-1}(y) \in X_A$. By the equality $h(\sigma_A(z)) = y$ with (2.1), one has $h(z) \in \sigma_B^{-l_1(z)}(\sigma_B^{k_1(z)}(y)) = \sigma_B^{-l_1(z)}(\sigma_B^{K_1}(y))$ and

$$h(z) = (\mu_1(z), \dots, \mu_{l_1(z)}(z), y_{K_1+1}, y_{K_1+2}, \dots) \quad (6.4)$$

for some $(\mu_1(z), \dots, \mu_{l_1(z)}(z)) \in B_{l_1(z)}(X_B)$. Put $L_1 = \text{Max}\{l_1(z) \mid z = ih^{-1}(y), y \in X_B^{(i)}\}$. The set

$$W^{(i)} = \{(\mu_1(z), \dots, \mu_{l_1(z)}(z)) \in B_{l_1(z)}(X_B) \mid z = ih^{-1}(y), y \in X_B^{(i)}\}$$

of words is a finite subset of $W_{L_1}(X_B) = \cup_{j=0}^{L_1} B_j(X_B)$. For a word $\nu = (\nu_1, \dots, \nu_j) \in W^{(i)}$, put a clopen set in $X_B^{(i)}$

$$E_\nu^{(i)} = \{y \in X_B^{(i)} \mid \mu_1(z) = \nu_1, \dots, \mu_{l_1(z)}(z) = \nu_j, z = ih^{-1}(y)\}$$

and the projections

$$Q_\nu^{(i)} = \chi_{E_\nu^{(i)}} \quad \text{and} \quad P^{(i)} = \chi_{X_B^{(i)}}$$

in \mathcal{D}_B , where $\chi_{E_\nu^{(i)}}$ and $\chi_{X_B^{(i)}}$ denote the characteristic functions on X_B for the clopen sets $E_\nu^{(i)}$ and $X_B^{(i)}$ respectively. Since $X_B^{(i)}$ is a disjoint union $X_B^{(i)} = \cup_{\nu \in W^{(i)}} E_\nu^{(i)}$, we have

$$P^{(i)} = \sum_{\nu \in W^{(i)}} Q_\nu^{(i)}.$$

For $y \in X_B^{(i)}$ and $\nu \in W^{(i)}$, we have $y \in E_\nu^{(i)}$ if and only if $Q_\nu^{(i)} e_y^B = e_y^B$. By (6.4), we have

$$e_{h(ih^{-1}(y))}^B = \sum_{\nu \in W^{(i)}} S_\nu^B \sum_{\xi \in B_{K_1}(X_B)} S_\xi^{B*} Q_\nu^{(i)} e_y^B \quad \text{for } y \in X_B^{(i)}. \quad (6.5)$$

Hence

$$U_h S_i^A U_h^* e_y^B = \sum_{\nu \in W^{(i)}} \sum_{\xi \in B_{K_1}(X_B)} S_\nu^B S_\xi^{B*} Q_\nu^{(i)} e_y^B \quad \text{for } y \in X_B^{(i)}$$

so that

$$U_h S_i^A U_h^* = \sum_{\nu \in W^{(i)}} \sum_{\xi \in B_{K_1}(X_B)} S_\nu^B S_\xi^{B*} Q_\nu^{(i)}.$$

As $Q_\nu^{(i)} \in \mathcal{D}_B$, we have $\text{Ad}(U_h)(S_i^A) \in \mathcal{O}_B$ so that $\text{Ad}(U_h)(\mathcal{O}_A) \subset \mathcal{O}_B$. Since $U_h^* = U_{h^{-1}}$, we symmetrically have $\text{Ad}(U_h^*)(\mathcal{O}_B) \subset \mathcal{O}_A$ so that $\text{Ad}(U_h)(\mathcal{O}_A) = \mathcal{O}_B$. It is direct to see that $\text{Ad}(U_h)(f) = f \circ h^{-1}$ for $f \in \mathcal{D}_A$ from the definition $U_h e_x^A = e_{h(x)}^B, x \in X_A$ so that we have $\text{Ad}(U_h)(\mathcal{D}_A) = \mathcal{D}_B$.

We will next show that $\text{Ad}(U_h) \circ \rho_t^A = \text{Ad}(u_t^{b_2}) \circ \rho_t^B \circ \text{Ad}(U_h)$ for $t \in \mathbb{T}$. It follows that

$$\begin{aligned} (\text{Ad}(u_t^{b_2}) \circ \rho_t^B \circ \text{Ad}(U_h))(S_i^A) e_y^B &= u_t^{b_2} \rho_t^B (U_h S_i U_h^*) (u_t^{b_2})^* e_y^B \\ &= \sum_{\nu \in W^{(i)}} \sum_{\xi \in B_{K_1}(X_B)} u_t^{b_2} \rho_t^B (S_\nu^B S_\xi^{B*} Q_\nu^{(i)}) u_{-t}^{b_2} e_y^B. \end{aligned}$$

Since $Q_\nu^{(i)} e_y^B \neq 0$ if and only if $Q_\nu^{(i)} e_y^B = e_y^B$ and $\nu_1 = \mu_1(z), \dots, \nu_j = \mu_{l_1(z)}(z)$. For $y \in E_\nu^{(i)} \cap U_\xi$, we have $S_\nu^B S_\xi^{B*} Q_\nu^{(i)} e_y^B = e_{h(z)}^B = e_{h(ih^{-1}(y))}^B$ so that

$$\begin{aligned} &u_t^{b_2} \rho_t^B (S_\nu^B S_\xi^{B*} Q_\nu^{(i)}) u_{-t}^{b_2} e_y^B \\ &= \exp(2\pi\sqrt{-1}(|\nu| - |\xi|)t) u_t^{b_2} S_\nu^B S_\xi^{B*} Q_\nu^{(i)} u_{-t}^{b_2} e_y^B \\ &= \exp(2\pi\sqrt{-1}(|\nu| - |\xi| - b_2(y))t) u_t^{b_2} S_\nu^B S_\xi^{B*} Q_\nu^{(i)} e_y^B \\ &= \exp(2\pi\sqrt{-1}(l_1(z) - k_1(z) - b_2(y))t) u_t^{b_2} e_{h(ih^{-1}(y))}^B \\ &= \exp(2\pi\sqrt{-1}(l_1(z) - k_1(z) - b_2(y) + b_2(h(ih^{-1}(y))))t) e_{h(ih^{-1}(y))}^B. \end{aligned}$$

Lemma 6.2 ensures us the equality $b_2(y) - b_2(h(ih^{-1}(y))) = b_1(z) - b_1(\sigma_A(z))$ so that

$$l_1(z) - k_1(z) - b_2(y) + b_2(h(ih^{-1}(y))) = c_1(z) - (b_1(z) - b_1(\sigma_A(z))) = 1.$$

As $e_{h(ih^{-1}(y))}^B = U_h S_i^A U_h^* e_y^B$, we have

$$u_t^{b_2} \rho_t^B (S_\nu^B S_\xi^{B*} Q_\nu^{(i)}) u_{-t}^{b_2} e_y^B = \exp(2\pi\sqrt{-1}t) U_h S_i^A U_h^* e_y^B = \text{Ad}(U_h)(\rho_t^A(S_i^A)) e_y^B.$$

Hence we have

$$\begin{aligned} (\text{Ad}(u_t^{b_2}) \circ \rho_t^B \circ \text{Ad}(U_h))(S_i^A) e_y^B &= \sum_{\nu \in W^{(i)}} \sum_{\xi \in B_{K_1}(X_B)} u_t^{b_2} \rho_t^B (S_\nu^B S_\xi^{B*} Q_\nu^{(i)}) u_{-t}^{b_2} e_y^B \\ &= \text{Ad}(U_h)(\rho_t^A(S_i^A)) e_y^B. \end{aligned}$$

We thus have

$$(\text{Ad}(u_t^{b_2}) \circ \rho_t^B \circ \text{Ad}(U_h))(S_i^A) = \text{Ad}(U_h)(\rho_t^A(S_i^A))$$

and hence

$$\text{Ad}(u_t^{b_2}) \circ \rho_t^B \circ \text{Ad}(U_h) = \text{Ad}(U_h) \circ \rho_t^A \quad \text{for } t \in \mathbb{T}.$$

By setting $\Phi = \text{Ad}(U_h) : \mathcal{O}_A \rightarrow \mathcal{O}_B$, we have a desired cocycle conjugacy. \square

To prove the converse of the above proposition, we provide the following lemma.

Lemma 6.4. *For a unitary representation u of \mathbb{T} into \mathcal{D}_B , there exists a continuous function $f_0 \in C(X_B, \mathbb{Z})$ such that $u_t = \exp(2\pi\sqrt{-1}t f_0)$ for $t \in \mathbb{T}$.*

Proof. For a unitary representation u of \mathbb{T} into \mathcal{D}_B , there exists a $*$ -homomorphism φ^u from the group C^* -algebra $C^*(\mathbb{T})$ of \mathbb{T} to \mathcal{D}_B in a natural way. It induces a homomorphism $\varphi_*^u : K_0(C^*(\mathbb{T})) \rightarrow K_0(\mathcal{D}_B)$ on their K -groups. Let χ_{id} denote the identity representation $\chi_{\text{id}}(s) = s, s \in \mathbb{T}$ of \mathbb{T} . As $K_0(C^*(\mathbb{T})) = \oplus_{\chi \in \hat{\mathbb{T}}} \mathbb{Z}$ and $K_0(\mathcal{D}_B) = C(X_B, \mathbb{Z})$, by putting $f_0 = \varphi_*^u(\chi_{\text{id}}) \in C(X_B, \mathbb{Z})$, one has $u_t = \exp(2\pi\sqrt{-1}tf_0)$ for all $t \in \mathbb{T}$. \square

We thus have the converse of the above proposition in the following way.

Proposition 6.5. *If there exist a unitary representation u of \mathbb{T} into \mathcal{D}_B and an isomorphism $\Phi : \mathcal{O}_A \rightarrow \mathcal{O}_B$ such that $\Phi(\mathcal{D}_A) = \mathcal{D}_B$ and $\Phi \circ \rho_t^A = \text{Ad}(u_t) \circ \rho_t^B \circ \Phi$ for $t \in \mathbb{T}$, then $(X_A, \sigma_A) \underset{\text{SCOPE}}{\sim} (X_B, \sigma_B)$.*

Proof. For the unitary representation $u_t \in U(\mathcal{D}_B)$, one may take a continuous function $f_0 \in C(X_B, \mathbb{Z})$ such that $u_t = \exp(2\pi\sqrt{-1}tf_0), t \in \mathbb{T}$. Represent the algebras \mathcal{O}_A on \mathfrak{H}_A and \mathcal{O}_B on \mathfrak{H}_B by (6.3). Since the isomorphism $\Phi : \mathcal{O}_A \rightarrow \mathcal{O}_B$ satisfies $\Phi(\mathcal{D}_A) = \mathcal{D}_B$, the one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent by [9], so that there exists a continuous orbit homeomorphism $h : X_A \rightarrow X_B$ such that $\Phi = \text{Ad}(U_h)$, where $U_h : \mathfrak{H}_A \rightarrow \mathfrak{H}_B$ is the unitary defined by $U_h e_x^A = e_{h(x)}^B$ for $x \in X_A$. Let $k_1 : X_A \rightarrow \mathbb{Z}_+$ and $l_1 : X_B \rightarrow \mathbb{Z}_+$ be continuous functions satisfying (1.2). For $i = 1, \dots, N$ and $y \in X_B^{(i)}$ put $z = ih^{-1}(y) \in X_A$. As in the proof of the preceding proposition, one sees that

$$\begin{aligned} & (\text{Ad}(u_t) \circ \rho_t^B \circ \text{Ad}(U_h))(S_i^A) e_y^B \\ &= \exp(2\pi\sqrt{-1}(l_1(z) - k_1(z) - f_0(y) + f_0(h(ih^{-1}(y))))t) e_{h(ih^{-1}(y))}^B \end{aligned}$$

and

$$\text{Ad}(U_h)(\rho_t^A(S_i^A)) e_y^B = \exp(2\pi\sqrt{-1}t) e_{h(ih^{-1}(y))}^B.$$

Since $\text{Ad}(U_h) \circ \rho_t^A = \text{Ad}(u_t) \circ \rho_t^B \circ \text{Ad}(U_h)$ for $t \in \mathbb{T}$, it follows that

$$l_1(z) - k_1(z) - f_0(y) + f_0(h(ih^{-1}(y))) - 1 = 0.$$

By putting $b_1(z) = f_0(h(z))$, we have $l_1(z) - k_1(z) = 1 + b_1(z) - b_1(\sigma_A(z))$ so that $(X_A, \sigma_A) \underset{\text{SCOPE}}{\sim} (X_B, \sigma_B)$. \square

We note the following lemma.

Lemma 6.6. *Let $v_t \in U(\mathcal{O}_B), t \in \mathbb{T}$ be a one-cocycle for the gauge action ρ^B on \mathcal{O}_B . If there exists an isomorphism $\Psi : \mathcal{O}_A \rightarrow \mathcal{O}_B$ such that $\Psi(\mathcal{D}_A) = \mathcal{D}_B$ and $\Psi \circ \rho_t^A = \text{Ad}(v_t) \circ \rho_t^B \circ \Psi$ for $t \in \mathbb{T}$, then v_t belongs to \mathcal{D}_B and hence $v_{t+s} = v_t v_s, t, s \in \mathbb{T}$.*

Proof. For $f \in \mathcal{D}_A$, we have $\Psi(\rho_t^A(f)) = v_t(\rho_t^B(\Psi(f)))v_t^*$. As $\rho_t^A(f) = f$ and $\rho_t^B(\Psi(f)) = \Psi(f)$, we see that $\Psi(f)v_t = v_t\Psi(f)$. Since $\Psi(\mathcal{D}_A) = \mathcal{D}_B$ and \mathcal{D}_B is a maximal commutative C^* -subalgebra of \mathcal{O}_B , the unitary v_t belongs to \mathcal{D}_B . \square

Consequently we have the following theorem.

Theorem 6.7. *The following two assertions are equivalent.*

- (i) *One-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are strongly continuous orbit equivalent.*
- (ii) *There exist a unitary one-cocycle $v_t \in \mathcal{O}_B, t \in \mathbb{T}$ for the gauge action ρ^B on \mathcal{O}_B and an isomorphism $\Phi : \mathcal{O}_A \rightarrow \mathcal{O}_B$ such that*

$$\Phi(\mathcal{D}_A) = \mathcal{D}_B \quad \text{and} \quad \Phi \circ \rho_t^A = \text{Ad}(v_t) \circ \rho_t^B \circ \Phi, \quad t \in \mathbb{T}.$$

As it is well-known that a cocycle conjugate covariant system of a locally compact abelian group yields a conjugate dual covariant system, we have the following corollary.

Corollary 6.8. *Assume that $(X_A, \sigma_A) \underset{\text{SCOE}}{\sim} (X_B, \sigma_B)$. Then the dual actions of their C^* -crossed products are isomorphic:*

$$(\mathcal{O}_A \rtimes_{\rho^A} \mathbb{T}, \hat{\rho}^A, \mathbb{Z}) \cong (\mathcal{O}_B \rtimes_{\rho^B} \mathbb{T}, \hat{\rho}^B, \mathbb{Z}).$$

7 Examples

1. Let A and B be the following matrices:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}. \quad (7.1)$$

They are both irreducible and satisfy condition (I). The one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent as in [9]. This continuous orbit equivalence also comes from the fact that their Cuntz-Krieger algebras \mathcal{O}_A and \mathcal{O}_B are isomorphic and $\det(\text{id} - A) = \det(\text{id} - B)$ by [12]. Since their Perron eigenvalues of A and of B are different, the topological entropy of the two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are different so that they are not topologically conjugate as two-sided subshifts.

2. If one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are topologically conjugate, one may take a homeomorphism $h : X_A \rightarrow X_B$ such that $k_1(x) = 0, l_1(x) = 1$ for all $x \in X_A$, so that $c_1(x) = 1$ for all $x \in X_A$ and hence (X_A, σ_A) and (X_B, σ_B) are strongly continuous orbit equivalent. We will present an example of one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) such that they are not topologically conjugate, but they are strongly continuous orbit equivalent. Let A and B be the following matrices:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}. \quad (7.2)$$

They are both irreducible and satisfy condition (I). Since the total column amalgamation of B is it-self, their one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are not topologically conjugate ([7], [19]). We will show the following theorem.

Theorem 7.1. *The one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are strongly continuous orbit equivalent.*

We will prove Theorem 7.1 as follows. Let us denote by $\Sigma_A = \{\alpha, \beta\}$ the symbols of the shift space X_A , and similarly $\Sigma_B = \{1, 2, 3\}$ those of (X_B, σ_B) , respectively. We note that

$$\begin{aligned} B_2(X_A) &= \{(\alpha, \alpha), (\alpha, \beta), (\beta, \alpha), (\beta, \beta)\}, \\ B_2(X_B) &= \{(1, 1), (1, 2), (2, 1), (2, 3), (3, 1), (3, 3)\}. \end{aligned}$$

Define the block maps Φ and φ by

$$\begin{aligned} \Phi(\alpha, \alpha) &= (1, 1), & \Phi(\beta, \beta, \alpha) &= (2, 1), & \Phi(\beta, \alpha, \alpha) &= (3, 1), \\ \Phi(\alpha, \beta) &= (1, 2), & \Phi(\beta, \beta, \beta) &= (2, 3), & \Phi(\beta, \alpha, \beta) &= (3, 3) \end{aligned}$$

and

$$\varphi(\alpha, \beta) = 2, \quad \varphi(\beta, \beta) = 3, \quad \varphi(\alpha, \alpha) = 1, \quad \varphi(\beta, \alpha) = 1.$$

It is direct to see that the 2-block map $\varphi : B_2(X_A) \rightarrow B_1(X_B)$ gives rise to a sliding block code from X_A to X_B . For $k, l \in \mathbb{Z}_+$, we denote by $\varphi_\infty^{[-k, l]}$ the sliding block code with memory k and anticipation l (see [8]). Define $h : X_A \rightarrow X_B$ by setting for $x = (x_n)_{n \in \mathbb{N}} \in X_A$

$$\begin{aligned} h(x_1, x_2, x_3, \dots) &= \begin{cases} (\Phi(x_1, x_2), \varphi_\infty^{[0, 1]}(x_2, x_3, \dots)) & \text{if } x_1 = \alpha, \\ (\Phi(x_1, x_2, x_3), \varphi_\infty^{[-1, 0]}(x_3, x_4, \dots)) & \text{if } x_1 = \beta \end{cases} \\ &= \begin{cases} (1, 1, \varphi_\infty^{[0, 1]}(\sigma_A(x))) & \text{if } (x_1, x_2) = (\alpha, \alpha), \\ (1, 2, \varphi_\infty^{[0, 1]}(\sigma_A(x))) & \text{if } (x_1, x_2) = (\alpha, \beta), \\ (2, 1, \varphi_\infty^{[-1, 0]}(\sigma_A^2(x))) & \text{if } (x_1, x_2, x_3) = (\beta, \beta, \alpha), \\ (2, 3, \varphi_\infty^{[-1, 0]}(\sigma_A^2(x))) & \text{if } (x_1, x_2, x_3) = (\beta, \beta, \beta), \\ (3, 1, \varphi_\infty^{[-1, 0]}(\sigma_A^2(x))) & \text{if } (x_1, x_2, x_3) = (\beta, \alpha, \alpha), \\ (3, 3, \varphi_\infty^{[-1, 0]}(\sigma_A^2(x))) & \text{if } (x_1, x_2, x_3) = (\beta, \alpha, \beta). \end{cases} \end{aligned}$$

We note that $h(x)$ belongs to X_B for all $x \in X_A$ because of the following equalities, where $h(x)_{[1, 3]}$ denotes the first three symbols of $h(x)$.

$$h(x)_{[1, 3]} = \begin{cases} (1, 1, 1) & \text{if } (x_1, x_2, x_3) = (\alpha, \alpha, \alpha), \\ (1, 1, 2) & \text{if } (x_1, x_2, x_3) = (\alpha, \alpha, \beta), \\ (1, 2, 1) & \text{if } (x_1, x_2, x_3) = (\alpha, \beta, \alpha), \\ (1, 2, 3) & \text{if } (x_1, x_2, x_3) = (\alpha, \beta, \beta), \\ (2, 1, 1) & \text{if } (x_1, x_2, x_3, x_4) = (\beta, \beta, \alpha, \alpha), \\ (2, 1, 2) & \text{if } (x_1, x_2, x_3, x_4) = (\beta, \beta, \alpha, \beta), \\ (2, 3, 1) & \text{if } (x_1, x_2, x_3, x_4) = (\beta, \beta, \beta, \alpha), \\ (2, 3, 3) & \text{if } (x_1, x_2, x_3, x_4) = (\beta, \beta, \beta, \beta), \\ (3, 1, 1) & \text{if } (x_1, x_2, x_3, x_4) = (\beta, \alpha, \alpha, \alpha), \\ (3, 1, 2) & \text{if } (x_1, x_2, x_3, x_4) = (\beta, \alpha, \alpha, \beta), \\ (3, 3, 1) & \text{if } (x_1, x_2, x_3, x_4) = (\beta, \alpha, \beta, \alpha), \\ (3, 3, 3) & \text{if } (x_1, x_2, x_3, x_4) = (\beta, \alpha, \beta, \beta). \end{cases}$$

We set

$$l_1(x) = \begin{cases} 1 & \text{if } (x_1, x_2) = (\alpha, \alpha), \\ 4 & \text{if } (x_1, x_2) = (\alpha, \beta), \\ 2 & \text{if } (x_1, x_2) = (\beta, \alpha), \\ 3 & \text{if } (x_1, x_2) = (\beta, \beta), \end{cases} \quad k_1(x) = \begin{cases} 0 & \text{if } (x_1, x_2) = (\alpha, \alpha), \\ 2 & \text{if } (x_1, x_2) = (\alpha, \beta), \\ 2 & \text{if } (x_1, x_2) = (\beta, \alpha), \\ 2 & \text{if } (x_1, x_2) = (\beta, \beta) \end{cases}$$

so that we have

$$\sigma_B^{k_1(x)}(h(\sigma_A(x))) = \sigma_B^{l_1(x)}(h(x)) \quad \text{for } x \in X_A$$

and

$$c_1(x) = \begin{cases} 1 & \text{if } (x_1, x_2) = (\alpha, \alpha), \\ 2 & \text{if } (x_1, x_2) = (\alpha, \beta), \\ 0 & \text{if } (x_1, x_2) = (\beta, \alpha), \\ 1 & \text{if } (x_1, x_2) = (\beta, \beta). \end{cases}$$

Define a continuous function $b_1 : X_A \rightarrow \mathbb{N}$ by

$$b_1(x) = \begin{cases} 2 & \text{if } x_1 = \alpha, \\ 1 & \text{if } x_1 = \beta. \end{cases}$$

Since

$$b_1(x) - b_1(\sigma_A(x)) = \begin{cases} 0 = c_1(x) - 1 & \text{if } (x_1, x_2) = (\alpha, \alpha), \\ 1 = c_1(x) - 1 & \text{if } (x_1, x_2) = (\alpha, \beta), \\ -1 = c_1(x) - 1 & \text{if } (x_1, x_2) = (\beta, \alpha), \\ 0 = c_1(x) - 1 & \text{if } (x_1, x_2) = (\beta, \beta), \end{cases}$$

we have

$$c_1(x) = 1 + b_1(x) - b_1(\sigma_A(x)), \quad x \in X_A.$$

This implies the following lemma.

Lemma 7.2. $h : X_A \rightarrow X_B$ is a strongly continuous orbit map.

We will next construct the inverse of h . Define the block maps Ψ and ψ by

$$\begin{aligned} \Psi(1, 1) &= (\alpha, \alpha), & \Psi(2, 1) &= (\beta, \beta, \alpha), & \Psi(3, 1) &= (\beta, \alpha, \alpha), \\ \Psi(1, 2) &= (\alpha, \beta), & \Psi(2, 3) &= (\beta, \beta, \beta), & \Psi(3, 3) &= (\beta, \alpha, \beta) \end{aligned}$$

and

$$\psi(1) = \alpha, \quad \psi(2) = \beta, \quad \psi(3) = \beta.$$

It is direct to see that the 1-block map $\psi : B_1(X_A) \rightarrow B_1(X_B)$ gives rise to a sliding block

code from X_B to X_A . Define $g : X_B \rightarrow X_A$ by setting for $y = (y_n)_{n \in \mathbb{N}} \in X_B$

$$g(y_1, y_2, y_3, y_4, \dots) = \begin{cases} (\Psi(y_1, y_2), \psi_{\infty}^{[0,0]}(y_3, y_4, y_5, \dots)) & \text{if } y_1 = 1, \\ (\Psi(y_1, y_2), \psi_{\infty}^{[-1,-1]}(y_3, y_4, y_5, \dots)) & \text{if } y_1 = 2, 3 \end{cases}$$

$$= \begin{cases} (\alpha, \alpha, \psi_{\infty}^{[0,0]}(\sigma_B^2(y))) & \text{if } (y_1, y_2) = (1, 1), \\ (\alpha, \beta, \psi_{\infty}^{[0,0]}(\sigma_B^2(y))) & \text{if } (y_1, y_2) = (1, 2), \\ (\beta, \beta, \alpha, \psi_{\infty}^{[-1,-1]}(\sigma_B^2(y))) & \text{if } (y_1, y_2) = (2, 1), \\ (\beta, \beta, \beta, \psi_{\infty}^{[-1,-1]}(\sigma_B^2(y))) & \text{if } (y_1, y_2) = (2, 3), \\ (\beta, \alpha, \alpha, \psi_{\infty}^{[-1,-1]}(\sigma_B^2(y))) & \text{if } (y_1, y_2) = (3, 1), \\ (\beta, \alpha, \beta, \psi_{\infty}^{[-1,-1]}(\sigma_B^2(y))) & \text{if } (y_1, y_2) = (3, 3). \end{cases}$$

We set

$$l_2(y) = \begin{cases} 3 & \text{if } (y_1, y_2) = (1, 1), (1, 2), \\ 4 & \text{if } (y_1, y_2) = (2, 1), (2, 3), (3, 1), (3, 3), \end{cases}$$

$$k_2(y) = \begin{cases} 2 & \text{if } (y_1, y_2) = (1, 1), (2, 1), (3, 1), \\ 3 & \text{if } (y_1, y_2) = (1, 2), (2, 3), (3, 3), \end{cases}$$

so that we have

$$\sigma_A^{k_2(y)}(g(\sigma_B(y))) = \sigma_A^{l_2(y)}(g(y)) \quad \text{for } y \in X_B$$

and

$$c_2(y) = \begin{cases} 1 & \text{if } (y_1, y_2) = (1, 1), (2, 3), (3, 3), \\ 0 & \text{if } (y_1, y_2) = (1, 2), \\ 2 & \text{if } (y_1, y_2) = (2, 1), (3, 1). \end{cases}$$

Define a continuous function $b_2 : X_B \rightarrow \mathbb{N}$ by

$$b_2(y) = \begin{cases} 1 & \text{if } y_1 = 1, \\ 2 & \text{if } y_1 = 2, 3. \end{cases}$$

Since

$$b_2(y) - b_2(\sigma_B(y)) = \begin{cases} 0 = c_2(y) - 1 & \text{if } (y_1, y_2) = (1, 1), \\ -1 = c_2(y) - 1 & \text{if } (y_1, y_2) = (1, 2), \\ 1 = c_2(y) - 1 & \text{if } (y_1, y_2) = (2, 1), (3, 1), \\ 0 = c_2(y) - 1 & \text{if } (y_1, y_2) = (2, 3), (3, 3), \end{cases}$$

we have

$$c_2(y) = 1 + b_2(y) - b_2(\sigma_B(y)), \quad y \in X_B.$$

This implies the following lemma.

Lemma 7.3. $g : X_B \rightarrow X_A$ is a strongly continuous orbit map.

We will next show that g, h are inverses to each other.

For $x_1 = \alpha$, we see

$$\Psi(\Phi(\alpha, x_2)) = \begin{cases} \Psi(1, 1) = (\alpha, \alpha) & \text{if } x_2 = \alpha, \\ \Psi(1, 2) = (\alpha, \beta) & \text{if } x_2 = \beta \end{cases}$$

so that $\Psi(\Phi(x_1, x_2)) = (x_1, x_2)$.

For $x_1 = \beta$, we see

$$\Psi(\Phi(\beta, x_2, x_3)) = \begin{cases} \Psi(2, 1) = (\beta, \beta, \alpha) & \text{if } (x_2, x_3) = (\beta, \alpha), \\ \Psi(2, 3) = (\beta, \beta, \beta) & \text{if } (x_2, x_3) = (\beta, \beta), \\ \Psi(3, 1) = (\beta, \alpha, \alpha) & \text{if } (x_2, x_3) = (\alpha, \alpha), \\ \Psi(3, 3) = (\beta, \alpha, \beta) & \text{if } (x_2, x_3) = (\alpha, \beta) \end{cases}$$

so that $\Psi(\Phi(x_1, x_2, x_3)) = (x_1, x_2, x_3)$. It is easy to see that the equalities

$$\begin{aligned} \psi(\varphi(\alpha, x_1, x_2, \dots)) &= (x_1, x_2, \dots), \\ \psi(\varphi(\beta, x_1, x_2, \dots)) &= (x_1, x_2, \dots) \end{aligned}$$

hold so that $\psi \circ \varphi = \sigma_A$ on X_A .

Lemma 7.4. $g(h(x)) = x$ for $x \in X_A$.

Proof. It follows that

$$\begin{aligned} g(h(x_1, x_2, x_3, \dots)) &= \begin{cases} g(\Phi(x_1, x_2), \varphi(x_2, x_3, \dots)) & \text{if } x_1 = \alpha, \\ g(\Phi(x_1, x_2, x_3), \varphi(x_3, x_4, \dots)) & \text{if } x_1 = \beta \end{cases} \\ &= \begin{cases} (\Psi(\Phi(x_1, x_2)), \psi(\varphi(x_2, x_3, \dots))) & \text{if } x_1 = \alpha, \\ (\Psi(\Phi(x_1, x_2, x_3)), \psi(\varphi(x_3, x_4, \dots))) & \text{if } x_1 = \beta \end{cases} \\ &= \begin{cases} (x_1, x_2, \sigma_A(x_2, x_3, \dots)) & \text{if } x_1 = \alpha, \\ (x_1, x_2, x_3, \sigma_A(x_3, x_4, \dots)) & \text{if } x_1 = \beta \end{cases} \\ &= (x_1, x_2, x_3, x_4, \dots). \end{aligned}$$

□

We will finally prove that $h(g(y)) = y$ for all $y = (y_n)_{n \in \mathbb{N}} \in X_B$. It is direct to see that

$$\Phi(\Psi(y_1, y_2)) = (y_1, y_2) \quad \text{for } (y_1, y_2) \in B_2(X_B).$$

We have

$$\begin{aligned} \varphi(\alpha, \psi(y_3, y_4, \dots)) &= (y_3, y_4, \dots) & \text{if } y_2 = 1, \\ \varphi(\beta, \psi(y_3, y_4, \dots)) &= (y_3, y_4, \dots) & \text{if } y_2 = 2, 3. \end{aligned}$$

We set $g(y) = (x_n)_{n \in \mathbb{N}} \in X_A$ As

$$\begin{aligned} (x_1, x_2) &= \begin{cases} (\alpha, \alpha) & \text{if } (y_1, y_2) = (1, 1), \\ (\alpha, \beta) & \text{if } (y_1, y_2) = (1, 2), \end{cases} \\ (x_1, x_2, x_3) &= \begin{cases} (\beta, \beta, \alpha) & \text{if } (y_1, y_2) = (2, 1), \\ (\beta, \beta, \beta) & \text{if } (y_1, y_2) = (2, 3), \\ (\beta, \alpha, \alpha) & \text{if } (y_1, y_2) = (3, 1), \\ (\beta, \alpha, \beta) & \text{if } (y_1, y_2) = (3, 3), \end{cases} \end{aligned}$$

we have

$$\begin{aligned} h(g(y)) &= h(\Psi(y_1, y_2), \psi(y_3, y_4, \dots)) \\ &= \begin{cases} (\Phi(\Psi(y_1, y_2)), \varphi(x_2, \psi(y_3, y_4, \dots))) & \text{if } (y_1, y_2) = (1, 1), (1, 2), \\ (\Phi(\Psi(y_1, y_2)), \varphi(x_3, \psi(y_3, y_4, \dots))) & \text{if } (y_1, y_2) = (2, 1), (2, 3), (3, 1), (3, 3) \end{cases} \\ &= \begin{cases} (\Phi(\Psi(y_1, y_2)), \varphi(\alpha, \psi(y_3, y_4, \dots))) & \text{if } y_2 = 1, \\ (\Phi(\Psi(y_1, y_2)), \varphi(\beta, \psi(y_3, y_4, \dots))) & \text{if } y_2 = 2, 3 \end{cases} \\ &= (y_1, y_2, y_3, y_4, \dots). \end{aligned}$$

Hence $h(g(y)) = y$ for all $y \in X_B$ so that $g = h^{-1}$, and hence $(X_A, \sigma_A) \underset{\text{SCOE}}{\sim} (X_B, \sigma_B)$.

3. We will finally present an example of two irreducible matrices with entries in $\{0, 1\}$ whose two-sided topological Markov shifts are topologically conjugate, but whose one-sided topological Markov shifts are not strongly continuous orbit equivalent. Let A and B be the following matrices:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = A^t = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}. \quad (7.3)$$

They are both irreducible and satisfy condition (I). Since the row amalgamation of A and the column amalgamation of B are both $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$, the two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate (cf. [7]). We however know that $\mathcal{O}_A \cong \mathcal{O}_3$ and $\mathcal{O}_B \cong \mathcal{O}_3 \otimes M_2$ ([3]). Hence their Cuntz–Krieger algebras are not isomorphic so that the one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are not continuously orbit equivalent.

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